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A Classification of Flag-transitive Classical $c.C_2$-geometries by Means of Generators and Relations

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A readable and uniform proof of the classification of the flag-transitive classical $c.C_2$-geometries with $s \geq 3$ is provided, by means of generators and relations. As by-products, explicit presentations of $U_5(2)$, McL and a triple cover of $U_4(3)$ are obtained.

1. INTRODUCTION

The aim of this paper is to provide a readable and largely self-contained proof of the classification of flag-transitive classical $c.C_2$-geometries by a unified method; that is, generators and relations. As by-products, we will have explicit presentations for the groups $U_5(2)$, McL and a triple cover of $U_4(3)$. The approach by generators and relations has already been adopted in [29], where explicit presentations for Suz and $\mathrm{Aut}(HS)$ were obtained. The present paper can be thought of a fully extended version of [29] to the flag-transitive classical $c.C_2$-geometries of rank 3 with $s \geq 3$.

We repeat the definition of classical $c.C_2$-geometries. A residually connected geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{Q}; *)$ of rank 3 is called a $c.C_2$-geometry if it satisfies the following conditions:

(1) The residues of elements of $\mathcal{P}$ (called *points*) are generalized quadrangles.
(2) The residues of elements of $\mathcal{L}$ (called *lines*) are generalized digons.
(3) The residues of elements of $\mathcal{Q}$ (called *planes*) are isomorphic to the geometry of vertices and edges of a complete graph.

If Γ admits a flag-transitive automorphism group on Γ, Γ is called *flag-transitive.* The *order* (s, t) of a flag-transitive $c.C_2$-geometry Γ is the order of the generalized quadrangle Γ_a for a point a. A $c.C_2$-geometry Γ is called *classical* if the residues of points are classical generalized quadrangles (for the precise definition, see Section 2), which are so far the unique known infinite families of finite generalized quadrangles admitting flag-transitive automorphism groups [13, p. 98]. Note that we define a classical generalized quadrangle to be thick; that is, $s \geq 2$ and $t \geq 2$.

We usually denote by (a, r, u) a typical maximal flag of Γ with $a \in \mathcal{P}$, $r \in \mathcal{L}$ and $u \in \mathcal{Q}$. For a flag-transitive automorphism group G of Γ and $x = a, r, u$, we denote by G_x and K_x the stabilizer of x in G and the kernel of the action of G_x on the residue Γ_x, respectively. Two points are called *collinear* if they are incident with a plane (line) in common. The *collinearity* graph of Γ is the graph defined on the set of points by declaring that two points lie on an edge whenever they are collinear.

In this paper, we will give an explicit proof of the following statement, where we use the ATLAS notation [5] and the standard notation (see e.g., [19]) to denote the isomorphism types of groups and classical generalized quadrangles, respectively (see also Section 2).

THEOREM. *Let Γ be a classical $c.C_2$-geometry of order (s, t) with $s \geq 3$, admitting a flag-transitive automorphism group G. Then one of the following holds:*
(a) *Γ is isomorphic to the geometry on 176 points with $G' \simeq U_5(2)$ and $\Gamma_a \simeq W(3)$.*
(b) *Γ is isomorphic to the geometry on 275 points with $G' \simeq McL$ and $\Gamma_a \simeq Q^-(5, 3)$.*

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(c) *Γ is isomorphic to either the geometry on 378 points with $G' \simeq 3 \cdot U_4(3)$ or its quotient geometry on 126 points with $G' \simeq U_4(3)$ by the free action of the center of order 3. In each case, $\Gamma_a \simeq H(3, 2^2)$.*
(d) *The residue Γ_a is isomorphic to $H(3, 3^2)$.*

The geometries Γ in case (d) are completely classified in [29]. They are isomorphic to either the geometry with $\mathrm{Aut}(\Gamma) \simeq Suz.2$ on 22 880 points or the geometry with $\mathrm{Aut}(\Gamma) \simeq HS.2$ on 1100 points. Although the method used in this paper also works in the case $s = 2$, I will not include that case in this paper, because several papers are now available for classifying such geometries without assuming flag-transitivity. (See [2], [4], [6]. For results, see [18].)

Now I explain the method of generators and relations, which used in this paper to characterize flag-transitive $c.C_2$-geometries with point residues isomorphic to a specified quadrangle. This method has been used by many authors (e.g. [24], [10], [29], [28]) in order to effectively classify flag-transitive geometries with specified local structures, although it relies heavily on coset enumerations by computers. It may be considered as a practical variant of a much more abstract approach to the classification, called the method of amalgams (see [12], [27], [15], [11]).

First, we will briefly review this abstract approach. Let M be a flag-transitive automorphism group acting on a geometry Δ of rank 3 with specified local structures, and let (a, r, u) be a maximal flag of Δ. The method of amalgams requires us first to classify the possible isomorphism classes of stabilizers M_a, M_r and M_u, together with the embeddings of their intersections into these groups (that is, the amalgam of these stabilizers). For each isomorphism class (M_a, M_r, M_u) of amalgams, we construct its amalgamated sum $\tilde{M}$ and the group geometry $\tilde{\Delta}$ determined by $\tilde{M}$ and its subgroups M_a, M_r and M_u. (Note that the isomorphism class of $\tilde{\Delta}$ is uniquely determined by the isomorphism class of amalgams.) There is a natural 2-covering from the geometry $\tilde{\Delta}$ onto Δ, which turns out to be universal (see, e.g., [27, Cor. 8]). Thus, if we may explicitly construct a simply connected geometry Δ_0, admitting a flag-transitive group M_0 with specified isomorphism class of amalgams, then $\Delta_0 \simeq \tilde{\Delta}$ and we may classify our geometry Δ by considering possible quotients of Δ_0.

This method is computer free and makes clear the relation between the isomorphism class of simply connected flag-transitive geometries with specified local structures and the isomorphism classes of amalgams of certain groups. However, it does not provide any method for finding explicit geometries admitting a flag-transitive group with an amalgam of stabilizers isomorphic to a given one. We can get rid of this difficulty by asking computers to calculate the index $|\tilde{M}: M_a|$ just after we obtain the list of possible amalgams. The method of generators and relations is a variant of the method of amalgams, stressing on this computation, coset enumerations. It is very effective if we have some theoretical bound of the indices, and it is not so large to be handled by computers. (Note that we are in this situation by the diameter bound for the $c.C_2$-geometries [3].) By coset enumerations, we can also find the universal 2-cover of the known geometry (see below), while the construction of the universal 2-cover of each geometry by hand is, in general, straightforward but not simple at all (see, e.g., [20], [17]). Thus the method of generators and relations saves much effort when we have to treat many cases, as in this paper (or in [28]).

The method could be explained in a general form as follows. As the starting point, we have to determine the exact isomorphism type of the stabilizer M_a of a point a and a presentation $\mathscr{R}(a)$ of M_a among generators $I = \{g_1, \ldots, g_m\}$ such that for $x = r, u$ we have $M_a \cap M_x = \langle I(x) \rangle$ for a subset $I(x)$ of I. We denote by N_x the kernel of the action of M_x on the residue at x $(x = r, u)$. We could derive some information about the groups M_r/N_r and M_u/N_u (but not necessarily the exact isomorphism classes of both of them), since they are flag-transitive on the corresponding residues. We have at least an upper bound of $|M_x|$ for $x=r, u$, since we have M_a, and therefore its subgroups N_x. For $x=r, u$, we take generators $J(x)$ of M_x including $I(x)$ and find some relations $\mathcal{R}(x)$ that are possibly satisfied with them. (Since we have some information about M_x/N_x, we obtain such relations modulo N_x. Then, by considering the action of $M_a \cap M_x$ on N_x ($\subseteq M_a$), we may have several candidates for such relations.) We can choose $\mathcal{R}(x)$ so that the group generated by $J(x)$ with presentation $\mathcal{R}(x)$ is of order at most $|M_x|$. Note that we have, in general, several choices of such $J(x)$ and $\mathcal{R}(x)$.

By adding possible generators $J(x)$ and relations $\mathcal{R}(x)$ $(x=r, u)$ to $\mathcal{R}(a)$, we have generators of M and sets $\mathcal{R}^i$ $(i=1, \ldots, n)$ of relations with the following property. For any flag-transitive geometry Δ with specified local structures and any flag-transitive automorphism group M of Δ, there is at least one $\mathcal{R}^i$ satisfied in M. (Up to here, we determined some sets of amalgams containing the complete representatives of isomorphism classes.)

Let $\tilde{M}^i$ be the group with presentation $\mathcal{R}^i$, and let $\tilde{M}^i_x$ (resp. $\tilde{N}^i_x$) be the subgroup of $\tilde{M}^i$ generated by the elements of $\tilde{M}^i$ corresponding to the generators of M_x (resp. N_x) for $x=a, r, u$. Then either $\tilde{M}^i=1$ (that is, there is no subgroup M satisfying the relations $\mathcal{R}^i$) or there is an epimorphism from $\tilde{M}^i$ to M. Note that we may have $\tilde{M}^i \simeq \tilde{M}^j$ for some $i \neq j$. Since $\mathcal{R}(a)$ gives a presentation of M_a, we have $\tilde{M}^i_a \simeq M_a$ and $\tilde{N}^i_x \simeq N_x$ for any $x=r$, u and i with $\tilde{M}^i \neq 1$. Since we determine $\mathcal{R}^i$ so that $|\tilde{M}^i_x/\tilde{N}^i_x| \leq |M_x/N_x|$ for $x=r$, u, we have $\tilde{M}^i_x \simeq M_x$ for $x=r, u$ by comparing their orders. Thus, if $\tilde{M}^i \neq 1$, there is a 2-cover from the group geometry $\tilde{\Delta}^i$ determined by $\tilde{M}^i$ and its subgroups $\tilde{M}^i_x$ onto the group geometry Δ' determined by M and its subgroups M_x $(x=a, r, u)$. Since the flag transitivity of M on Δ implies that $\Delta' \simeq \Delta$, the group geometry $\tilde{\Delta}^i$ is a simply connected geometry with specified local structures. Hence we can find all simply connected geometries in question by determining all $\tilde{\Delta}_i$ with $\tilde{M}_i \neq 1$ and then by examining whether it satisfies specified local conditions.

Now we ask the computer to calculate the index $P_i := |\tilde{M}^i : \tilde{M}^i_a|$ by coset enumeration for each i. Note that P_i gives the number of points of a simply connected geometry $\tilde{\Delta}^i$ if $P_i \neq 1$. Assume that we have finite P_i for each i. We then discard i with $P_i=1$ and try to determine the isomorphism class of the group $\tilde{M}^i$ with $P_i \neq 1$. This is not so difficult to carry out, since we have the order $|\tilde{M}^i| = P_i\,|M_a|$ of $\tilde{M}^i$ and a subgroup M_a of $\tilde{M}^i$ of relatively small index. Once we have determined the group $\tilde{M}^i$, then the simply connected geometries $\tilde{\Delta}^i$ and isomorphisms among them can be easily obtained by examining subgroups of $\tilde{M}^i$'s.

If we do not know the universal cover of a known geometry with p points, we can find it among geometries $\tilde{\Delta}^i$ with P_i a multiple of p. For example, the simply connected $c.C_2$-geometry Γ with $\mathrm{Aut}(\Gamma) \simeq 3U_4(3) \cdot 2^2$ (see Section 6) was recognized by this method. This geometry was overlooked in [2], and also its simply-connectedness was not checked in the earliest version of [7]. That is, the method by generators and relations gives not only a shorter proof and a way to find correct answers, but also a proof of simply-connectedness by comparing known examples. Thus we may use computers not only to find all the answers but also to find universal covers of known geometries.

Hence, in this paper, I use coset enumerations.

Finally, I will describe the outline of this paper. In Section 2, we review several properties of classical generalized quadrangles and their automorphism groups, which are frequently used in this paper. In Section 3, we observe that the kernel K_a of a point a is quite small (see Lemma 7), and then restrict the possible isomorphism types of the residues Γ_a as well as the induced automorphism groups G_a/K_a (see Proposition 14). At this stage, we need a classification of flag-transitive subgroups of Chevalley groups [23] (see [13, C.7.1] for correction) and the transitive extension theorem by Suzuki [25]. In order to eliminate the case (N) (see Section 3, Lemma 6), we quote [7, 4.5 Case (3)], although we can avoid it by affording a similar proof. I also quote a result in [3], which gives a bound of the diameter of the collinearity graph of Γ. Although we can avoid it to classify our geometries, this theorem is very important. Indeed, it makes clear why our geometries can be classified in principle (see Proposition 9, Theorem 12). Furthermore, we can also use it to show the simply-connectedness of the geometries (a), (b), (c) in the theorem.

Up to here, we have worked along with the lines of [2, 7]. In Lemma 13, all geometries Γ, except those with $\Gamma_a \simeq W(3)$, $Q^-(5, 3)$, $H(3, 2^2)$ or $H(3, 3^2)$, will be eliminated by observing the structures of parabolic subgroups admitting transitive extensions. This argument is very elementary and unified, but seems to be new. The corresponding eliminations have done in [§2, 7] by much more combinatorial and *ad hoc* arguments.

Since the geometries Γ with $\Gamma_a \simeq H(3, 3^2)$ are completely classified in [29] by the use of generators and relations, it suffices to consider the other three cases. In Sections 4, 5 and 6, flag-transitive classical $c.C_2$-geometries Γ with $\Gamma_a \simeq W(3)$, $Q^-(5, 3)$ and $H(3, 2^2)$ are classified, by applying the method of generators and relations as follows.

In Section 3, the isomorphism types of a point-stabilizer G_a have been almost determined. Since $X_a/K_a := (G_a/K_a)'$ is a simple Chevalley group, we can write down a presentation for X_a by modifying the Steinberg relations. In particular, we have explicit generators of $X_a \cap G_r$ and $X_a \cap G_u$ for a maximal flag (a, r, u). Then we try to extend $X_u \cap G_u$ to a normal subgroup X_u of G_u of small index, by adjoining an element v which interchanges the two points incident with the line r. We can show that the subgroup $X := \langle X_a, X_u \rangle = \langle X_a, v \rangle$ is a normal subgroup of G of small index. (Thus we apply the method of generators and relations to this group X by taking as $\mathcal{R}(a)$ above the set of modified Steinberg relations.) In Sections 4–6, the work is devoted to seeking the possible sets of relations between v and the generators of X_a. Since X_u/K_u is proved to be isomorphic to the alternating or symmetric group, we easily have a presentation for X_u/K_u, involving a generator v. By examining the action of this group on N_u, we can show that there is a unique possible set $\mathcal{R}$ of relations in each case. (Note that we do not care so much about $X \cap G_r$.) Thus our situation is quite simple.

Let $\tilde{X}$ be the abstract group with generators those for X_a and v defined by $\mathcal{R}$, and let $\tilde{X}_x$ be the subgroup of $\tilde{X}$ corresponding to X_x for $x = a, r, u$. Then there is an epimorphism ρ from $\tilde{X}$ onto X. The restriction of ρ on $\tilde{X}_a$ is an isomorphism, since $\mathcal{R}$ contains the presentation of the group X_a. Now I calculate $|\tilde{X} : \tilde{X}_a|$ by coset enumeration using SUN/UNIX Cayley V. 3.7. In Sections 4–6, if $\Gamma_a \simeq W(3)$, $Q^-(5, 3)$ and $H(3, 2^2)$, we will show that $|\tilde{X} : \tilde{X}_a| = 176$, 275 and 378, and therefore $\tilde{X} \simeq U_5(2)$, McL and $3U_4(3)$, respectively. Then the isomorphism classes of Γ can be determined by classifying all non-trivial quotients W of $\tilde{X}$ and then calculating conjugacy classes of triples (W_a, W_r, W_u) of subgroups of W with isomorphism from $\tilde{X}_x$ onto W_x compatible with intersections $(x = a, r, u)$.

Since we have only one possible set $\mathcal{R}$ of relations in each case, we could avoid coset enumerations, by constructing the universal 2-cover of each known geometry (see [20] for the geometries admitting McL and $U_5(2)$).

It should be mentioned that several approaches are also possible to classify our geometries. The first one is to patch together [7], [29] and part of [2], since it is shown in [7] that the classification can be reduced either to the case completely classified in [29] or to the case treated in [2]. However, these two papers adopted quite different methods: the arguments in [2] are mainly combinatorial and very successful to treat the case $s = 2$ [2, §2, 4]. For the case $s \geqslant 3$, the authors tried to reduce the problem to the classification of certain strongly regular graphs of rank 3 [2, §7]. However, some arguments should be modified a little: one possible case for $s = 9$ was overlooked, which yields the 3-local geometry for *Suz* of rank 6 [29], and the case ruled out in 7.7(4) is, on the contrary, realizable in the geometry for $3U_4(3)$ of rank 5 mentioned above. Furthermore, several non-trivial results such as [8] and [9] were used to characterize the resulting rank-3 graphs.

The second one is the recent work by Meixner [14]. In his paper, he classified almost all flag-transitive $c^m.C^n$-geometries (circular extensions of $c.C^n$-geometries). As one of the starting points of such towers, he gave a characterization of the geometry for $3U_4(3)$ by means of graph theory, along the lines of [2]. However, he does not give any characterization of geometries for *Suz* and Aut(*HS*) in the same style. It seems to me that much more work would be required if we tried to characterize them by constructing their collinearity graphs.

The third one is the approach following the method of amalgams by A. Pasini [17]. As explained above, this method is essentially the same as my approach, but he tries to avoid using computers or explicit relations. However, it seems to require more effort than ours.

2. Classical Generalized Quadrangles

An incidence geometry $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1; *)$ of rank 2 is called a *generalized quadrangle* of *order* (s, t), if for any vertex $x \in \mathscr{G}_0$ $(\mathscr{G}_1)$ there are exactly $t+1$ $(s+1)$ vertices of $\mathscr{G}_1$ ($\mathscr{G}_0$, respectively) incident with x and, for any non-incident pair (x_0, x_1) of vertices $x_i \in \mathscr{G}_i$ $(i = 0$ or $1)$, there are unique vertices $y_i \in \mathscr{G}_i$ $(i = 0$ or $1)$ such that $x_0 * y_1 * y_0 * x_1$. We call elements of $\mathscr{G}_0$ (or $\mathscr{G}_1$) *isotropic* points (or lines, respectively). The *incidence* graph $\Gamma(\mathscr{G})$ of $\mathscr{G}$ is a graph with the set of vertices $V := \mathscr{G}_0 \cup \mathscr{G}_1$ and the set of edges $\{\{x, y\} \mid x \neq y \in V, x * y\}$. Any circuit of length 8 of the incidence graph $\Gamma(\mathscr{G})$ is called an *apartment*, and any chain $\alpha = (x_0, y_1, x_2, y_3, x_4)$ of length 4 is called a *root* of $\mathscr{G}$ with *middle vertex* x_2 if it is contained in an apartment. For more general definitions of apartments and roots of buildings of spherical type, see [21, p. 31].

A generalized quadrangle $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1; *)$ of order (s, t) is called *classical* if it is thick (that is, $s, t \geqslant 2$) and isomorphic to one of the generalized quadrangles associated with the Chevalley groups of rank 2. Precisely, they consist of the following 5 families, where X-quadrangle $\mathscr{G}$ means that X is a simple Chevalley group of rank 2 contained in Aut($\mathscr{G}$) (we use ATLAS notation to denote isomorphism types of groups throughout this paper):

$S_4(q)$-quadrangle $W(q)$ of order (q, q),

$O_5(q)$-quadrangle $Q(4, q)$ of order (q, q),

$O_6^-(q)$-quadrangle $Q^-(5, q)$ of order (q, q^2),

$U_4(q)$-quadrangle $H(3, q^2)$ of order (q^2, q) and

$U_5(q)$-quadrangle $H(4, q^2)$ of order (q^2, q^3),

for any prime power q.

In this section, we will review some properties of these quadrangles and their automorphism groups. We set $S := (\mathrm{Aut}(\mathscr{G}))^\infty$, a simple Chevalley group associated with $\mathscr{G}$.

It is known that the quadrangle $W(q)$ is isomorphic to the dual of $Q(4, q)$ (this fact induces the isomorphism $S_4(q) \simeq O_5(q)$), and that $Q^-(5, q)$ is isomorphic to the dual of $H(3, q^2)$ (this fact induces the isomorphicm $O_6^-(q) \simeq U_4(q)$). Furthermore, $W(q) \simeq$

Q(4, q) if q is even [19, 3.2.1,3]. By a result of Tits, the automorphism group $\mathrm{Aut}(\mathscr{G})$ of a classical generalized quadrangle $\mathscr{G}$ is isomorphic to the group obtained by adjoining diagonal and field automorphisms to the associated simple Chevalley group. That is:

LEMMA 1. *For $q = p^f$ with p prime, the full automorphism groups of classical generalized quadrangles are given as follows, where $S := (\mathrm{Aut}(\mathscr{G}))^\infty$:*

$\mathscr{G}$	S	$\mathrm{Aut}(\mathscr{G})/S$
$W(q)$ *or* $Q(4, q)$	$S_4(q)$	$(2, q-1){:}f$
$H(3, q^2)$ *or* $Q^-(5, q)$	$U_4(q)$	$(4, q+1){:}\,2f$
$H(4, q^2)$	$U_5(q)$	$(5, q+1){:}\,2f$

A *symmetry* about x means an element of $\mathrm{Aut}(\mathscr{G})$ fixing all vertices distant at most 2 from x in the incidence graph of $\mathscr{G}$ (see [19, p. 165]). Take any root $\alpha = (x_0, y_1, x_2, y_3, x_4)$ of $\mathscr{G}$ with middle vertex $x = x_2$. A symmetry about x fixes all vertices on α and all vertices incident with vertices y_i ($i = 1, 3$) and x. Thus the group Z_x of symmetries about x is contained in the *root subgroup* of α (see [21, p. 66]). The root subgroup of α acts semi-regularly on the set of apartments containing α (see, e.g., a remark in [21, p. 66]). Since $\mathscr{G}$ is a classical generalized quadrangle, the root group of α acts regularly on the set of t (or s) apartments containing α with middle vertex $x \in \mathscr{L}$ (or $x \in \mathscr{Q}$, respectively). Furthermore, the root subgroup is contained in S. Thus the group Z_x of symmetries about x is a subgroup of S of order dividing s or t.

Now we fix a chamber (l, u) of $\mathscr{G}$ with $l \in \mathscr{G}_0$ and $u \in \mathscr{G}_1$. For $x = l$ or u, we use S_x, N_x and U_x to denote the stabilizer of x in S, the kernel of the action of S_x on Γ_x and $O_p(N_x)$, respectively, where $q = p^f$. Under this notation, we will summarize below several properties of parabolic subgroups of S, which can be verified by straightforward computations.

LEMMA 2. *For $\mathscr{G}$ isomorphic to the $S_4(q)$-quadrangle $W(q)$, we have the following (O.l) and (O.u) if q is odd, and (E.x) for both $x = l$ and u, if q is even:*
(O.l) The group U_l is a special group of order q^3, in which the center $Z(U_l)$ is an elementary abelian group of order q consisting of all symmetries about l. There are subgroups $L_l \simeq SL_2(q)$ and $D_l \simeq (q-1)/(2, q-1)$ with $[L_l, D_l] = 1$ and $L_l \cap D_l = Z(L_l)$. We have $S_l = U_l{:}(D_l * L_l)$ and $N_l = U_l D_l$, and so $S_l/N_l \simeq L_2(q)$. The group $U_l/Z(U_l)$ is a natural module for $L_l \simeq SL_2(q)$, admitting a fixed point free action of D_l.
(O.u) The group U_u is an elementary abelian group of order q^3. The group of symmetries about u is trivial. We have $S_u/N_u \simeq PGL_2(q)$.
(E.x) The group U_x is an elementary abelian group of order q^3, in which the symmetries about x form a S_x-invariant subgroup Z_x of order q. There are subgroups $L_x \simeq L_2(q)$ and $D_x \simeq (q-1)$ with $[L_x, D_x] = 1$ and $L_x \cap D_x = 1$. We have $S_x = U_x{:}(D_x \times L_x)$ and $N_x = U_x D_x$, and so $S_x/N_x \simeq L_2(q)$. The group U_x/Z_x is a natural module for $L_x \simeq SL_2(q)$, admitting a fixed point free action of D_x. Furthermore, the group L_x acts trivially on Z_x.

LEMMA 3. *If $\mathscr{G}$ is isomorphic to the $U_4(q)$-quadrangle $H(3, q^2)$, the following hold:*
(l) The group U_l is a special group of order q^5, in which the center $Z(U_l)$ is an elementary abelian group of order q consisting of all symmetries about l. There are subgroups L_l isomorphic to $SU_2(q)(\simeq SL_2(q))$ and D_l isomorphic to a cyclic group of order $(q^2-1)/(4, q+1)$ with $[L_l, E_l] = 1$ and $L_l \cap D_l = 1$, where E_l is the unique

subgroup of D_l of index $(2, q-1)$. We have $S_l = U_l:(D_l * L_l)$, $N_l = U_lE_l$, and $S_l/N_l \simeq PGL_2(q)$. The group $U_l/Z(U_l)$ is a natural module for $L_l \simeq SU_2(q)$, admitting a fixed point free action of D_l.
(u) The group U_u is an elementary abelian group of order q^4. The group of symmetries about u is trivial. We have $S_u/N_u \simeq PSL_2(q^2)$.

LEMMA 4. *If $\mathscr{G}$ is isomorphic to the $U_5(q)$-quadrangle $H(4, q^2)$, the following hold:*
(l) The group U_l is a special group of order q^7, in which the center $Z(U_l)$ is an elementary abelian group of order q consisting of all symmetries about l. We have $S_l/N_l \simeq PGU_3(q)$, and the group $U_l/Z(U_l)$ is a natural module for $PGU_3(q)$.
(u) The group U_u is a special group of order q^8 with center $Z(U_u)$ of order q^4. There are subgroups $L_u \simeq SL_2(q^2)$ and E_u isomorphic to the cyclic group of order $(q^2-1)/(5, q+1)$ with $[L_u, E_u]=1$ and $L_u \cap E_u = 1$. The group $U_l:(E_u * L_u)$ is of index $(2, q-1)$ in S_u. We have $N_u = U_uE_u$ and $S_u/N_u \simeq PGL_2(q^2)$. The group $U_u/Z(U_u)$ is a natural module for $L_u \simeq SL_2(q^2)$, admitting a fixed point free action of E_u.

3. POSSIBLE TYPES OF RESIDUES OF POINTS

Let $\Gamma = (\mathscr{P}, \mathscr{L}, \mathscr{Q}; *)$ be a classical $c.C_2$-geometry admitting a flag-transitive group G. We write $q = p^f$ for a prime p, where Γ_a is isomorphic to $W(q)$, $Q(4, q)$, $Q^-(5, q)$, $H(3, q^2)$ or $H(4, q^2)$. We fix a chamber (a, l, u) of Γ with $a \in \mathscr{P}$, $l \in \mathscr{L}$ and $u \in \mathscr{Q}$. For $x = a$, l or u, we use G_x and K_x to denote the stabilizer of x in G and the kernel of the stabilizer G_x on the residue Γ_x. We also set $X_a/K_a := (G_a/K_a)^\infty$.

Now we will start to limit the possibilities of the isomorphism classes of Γ_a. First, by the flag-transitivity of G, G_a/K_a is a flag-transitive subgroup of $\mathrm{Aut}(\Gamma_a)$. Thus, by a part of the theorem of Seitz on the flag-transitive subgroups of Chevalley groups (on buildings) [23; 13, C. 7.1, pp. 148–149], we have the following possibilities:

LEMMA 5. *One of the following holds:*
(G) The group X_a/K_a is isomorphic to the simple group $(\mathrm{Aut}(\Gamma_a))^\infty$.
(U) We have $\Gamma_a \simeq H(3, 3^2)$, and G_a/K_a is isomorphic to either $L_3(4).2_3$ or $L_3(4).(2^2)_{133}$.
(N) We have $\Gamma_a \simeq W(3)$ or $Q(4, 3)$, and G_a/K_a is isomorphic to $2^4{:}S_5$, $2^4{:}A_5$ or $2^4{:}F_5^4$.

By [7, 4.5], the case (N) does not occur. In the remaining cases (G) and (U), we can verify that the point stabilizer G_a acts primitively on the set $\mathscr{L}(a)$ of lines through a. For two lines l and l' through the point a, we define $l \sim l'$ whenever $\mathscr{P}(l) = \mathscr{P}(l')$. Then the equivalence classes of $\sim$ form imprimitivity blocks of G_a in $\mathscr{L}(a)$. Thus the primitivity of G_a on $\mathscr{L}(a)$ implies claim (2) of the followng lemma, as $|\mathscr{P}(l)| = 2$.

LEMMA 6. *The following holds:*
(1) The case (N) above does not occur.
(2) There is at most one line through two distinct points.

By Lemma 6 above, we may identify a line $l \in \mathscr{L}$ with a pair $\{a, b\}$ of points on l. We will do so from now on.

LEMMA 7. *The following holds for a point b collinear with a:*
(1) The image K_bK_a/K_a of K_b in G_a/K_a is contained in the group of symmetries about the line $l = \{a, b\}$ of Γ_a.
(2) We have $K_a \cap K_b = 1$ and K_a is a p-group of order dividing t. In particular, $\langle K_c \mid c \in \mathscr{P}(u)\rangle$ is an elementary abelian normal p-subgroup of K_u.
(3) If the group of symmetries about the line $l = \{a, b\}$ of Γ_a is trivial, then $K_a = 1$.

PROOF. These statements are essentially involved in Lemma 3 [29], but we will give an explicit proof.

(1) Since the kernel K_a fixes all the lines through a, it fixes all the points collinear with a. (Recall that two points are called collinear if there is a line through them.) Let b be any point collinear with a, and u any plane through the line $l = \{a, b\}$. Any point c on u is collinear with b, since Γ_u is a circle geometry. Thus K_b fixes the point c and the line $\{a, c\}$. This implies that the image K_bK_a/K_a of K_b in G_a/K_a is contained in the group of symmetries about the isotropic point $l = \{a, b\}$ of Γ_a.

(2) Let v be any plane through a line $l = \{a, b\}$ and c be any point on v different from a, b. The image $(K_a \cap K_b)K_c/K_c$ of $K_a \cap K_b$ in G_c/K_c is contained in the groups of symmetries about $\{a, c\}$ and $\{b, c\}$ by (1). Thus it fixes an apartment containing a root $(w, \{a, c\}, v, \{b, c\})$, and therefore it is the trivial group. Then we have $K_a \cap K_b \subseteq K_c$, and so $K_a \cap K_b = K_a \cap K_c = K_b \cap K_c$.

Applying this results in the line $\{b, c\}$, $K_a \cap K_b = K_c \cap K_d$ for any distinct points c, d on v. In particular, this group fixes any point on any plane through $\{c, d\}$. Since Γ is connected, any two planes are joined by a chain of planes in which any two subsequent planes are incident with a line in common. Thus, by repeating the above results, we conclude that $K_a \cap K_b$ fixes all the points and planes, and that $K_a \cap K_b = 1$. In particular, K_b is isomorphic to its image in G_a/K_a, which is a subgroup of symmetries about $\{a, b\}$ of order dividing t. Thus K_a is a p-group dividing t.

(3) By the assumption and (1), we have $K_bK_a/K_a = 1$, and therefore $K_a = K_b$. By (2), we have $K_a = K_b = 1$. □

The above lemma implies that the stabilizer G_a of a point is of finite order. On the other hand, the theorem of Cameron, Hughes and Pasini [3] implies that the diameter of the collinearity graph of Γ is bounded by $s+1$, where (s, t) is the order of the generalized quadrangle Γ_a. Thus, we have the following important result:

COROLLARY 8. *If Γ is a classical c.C_2-geometry admitting a flag-transitive group G, then G is a finite group and Γ is a finite geometry. Furthermore, for a given classical generalized quadrangle $\mathcal{G}$, there are finitely many such geometries Γ with residues of points isomorphic to $\mathcal{G}$.*

We may give an explicit bound for the number of points of Γ in terms of the order (s, t) of Γ_a, and therefore the bound for the order of G. This bound can be used to exclude a geometry associated with the Lyon's sporadic simple group (see [22, p. 84]) from our list of flag-transitive classical $c.C_2$-geometries.

Since Γ_u is a circle geometry and any line on u can be identified with a pair of points on it, the kernel K_u on the plane u consists of all elements of G fixing all points on u. Thus K_a is a normal subgroup of K_u ($\subseteq G_a$) and K_l is a normal subgroup of K_u ($\subseteq G_l$). The group K_a is also a normal subgroup of K_l. Since K_a is a p-group by Lemma 7(2), the kernel of the action of $X_a/K_a = (G_a/K_a)^\infty$ on the residue $\Gamma_{a,x}$ is given by $(X_a \cap G_x)/K_a$ for $x = u$ or l.

Now, in order to restrict the order (s, t), we will observe that G_u/K_u is a transitive extension of a doubly transitive group of special type. This fundamental and important observation has already been made in [2].

LEMMA 9. *The group G_u/K_u is a triply transitive permutation group on the set $\mathcal{P}(u)$ of $s+2$ points in which the stabilizer $A := (G_a \cap G_u)/K_u$ of a point a is a doubly transitive permutation group on the set $\Omega := \mathcal{P}(u)\backslash\{a\}$ of Suzuki type; that is, the stabilizer A_b of a point $b \in \Omega$ contains a characteristic subgroup $O_p(A_b)$ acting regularly on the set $\Omega\backslash\{b\}$ of s points.*

PROOF. The group X_a/K_a is a simple group $(\mathrm{Aut}(\Gamma_a))^\infty$ if the case (G) in Lemma 5 holds and $L_3(4)$ if the case (U) holds. In any case, by Lemmas 2–4 and Lemma 5, we may conclude that the stabilizer $X_a \cap G_u$ of u in X_u induces the permutation group $L_2(s)$ or $PGL_2(s)$ on the set Ω of $s+1$ points. Then $L := ((X_a \cap G_u)/K_a)' \simeq L_2(q)$ is a normal subgroup of A, as $X_a \cap G_u$ is normal in $G_{u,a}$. For a point $b \in \Omega$, the stabilizer L_b of b in L contains a normal subgroup Q acting regularly on the set $\Omega\backslash\{b\}$ of s points. It suffices to show that $O_p(A_b) = Q$.

The group $C_A(L)(\subseteq C_A(Q))$ fixes a unique Q-orbit $\{b\}$ on Ω of length 1. Since L is transitive on Ω, $C_A(L)$ fixes all the points of Ω, and so $C_A(L) = 1$ as a permutation group on $\mathscr{P}(u)$. In particular, A is a subgroup of $\mathrm{Aut}(L)$ containing L. Since $\mathrm{Aut}(L)$ is obtained by adjoining the diagonal and field automorphisms to $L_2(s)$, observing this explicit group we may verify that $O_p(A) = Q$. □

COROLLARY 10. *If the residue Γ_a is of order (s, t) with $s \geq 3$ and satisfies the condition in the case (G) in Lemma 5, then Γ_a is isomorphic to one of the following quadrangles: $W(3)$, $W(4)$, $Q(4, 3)$, $Q(4, 9)$, $Q^-(5, 3)$, $Q^-(5, 4)$, $H(3, 2^2)$, $H(3, 3^2)$ and $H(4, 2^2)$.*

PROOF. Since $(X_a \cap K_u)/K_u$ is not solvable for $s \geq 4$, it follows from [25, Theorem 5] that $s = 3$, 4 or 9, and that if $s = 9$, $G_u/K_u \simeq M_{11}$ and $G_{u,a}/K_u \simeq M_{10}$. In particular, $(X_a \cap G_u)/K_u$ does not contain $PGL_2(9)$. Then it follows from Lemmas 2–4 that Γ_a is not isomorphic to $W(9)$, $Q^-(5, 9)$ (dual of $H(3, 3^2)$) or $H(4, 3^2)$. That is, we have the list stated in the corollary as the remaining possibilities for the isomorphism classes of Γ_a. □

Corollaries 8 and 10 together yield the following important result, which makes us sure that a classification of the classical flag-transitive $c.C_2$-geometries is possible:

THEOREM 11. *There are only finitely many classical flag-transitive $c.C_2$-geometries.*

Furthermore, we will eliminate several of the possibilities listed in Corollary 10 for the residue of points, by observing the action of G_u on K_u.

LEMMA 12. *In the list of Corollary 10, the following possibilities do not occur: $W(4)$, $Q(4, 3)$, $Q(4, 9)$, $Q^-(5, 4)$ and $H(4, 2^2)$.*

PROOF. Suppose that Γ_a is isomorphic to one of the quadrangles in the lemma. We set $U_u := O_p(K_u)$, $U_{a,u} := O_p(X_a \cap K_u)$ and $Z_{a,u} :=$ the inverse image of $Z(U_{a,u}/K_a)$ $(\subseteq X_a/K_a)$ in X_a if $\Gamma_a \simeq W(4)$. If $\Gamma_a \simeq W(4)$, we let $Z_{a,u}$ be the group of elements of K_u fixing any plane v containing a and a point $b \in \mathscr{P}(u)$. Note that $K_a \subseteq Z_{a,u}$ and that $Z_{a,u}/K_a$ coincides with the group of symmetries about the isotropic line u in $\Gamma_a \simeq W(4)$.

We set $A_u/K_u := (G_u/K_u)'$. The stabilizer of a in G_u/K_u contains a normal subgroup $(X_a \cap G_u)/K_u$ isomorphic to $PGL_2(3) \simeq S_4$ if $\Gamma_a \simeq Q(4, 3)$, $L_2(9) \simeq A_6$ if $\Gamma_a \simeq Q(4, 9)$ and $L_2(4) \simeq A_5$ if $\Gamma_a \simeq Q^-(5, 4)$, $H(4, 2^2)$ or $W(4)$ (the cases with $p = 2$). Thus $A_u/K_u \simeq A_5$ (M_{11} or A_6) if $\Gamma_a \simeq Q(4, 3)$ ($Q(4, 9)$ or $p = 2$, respectively).

By Lemmas 2–4, there is a complement of $U_{a,u} = O_p((K_u \cap X_a)/K_a)$ in $(K_u \cap X_a)/K_a$. This group is a cyclic p'-group of order 1, 4, 3, 15 and 3, if $\Gamma_a \simeq Q(4, 3)$, $Q(4, 9)$, $Q^-(5, 4)$, $H(4, 2^2)$ and $W(4)$, respectively. Since K_a is a p-group by Lemma 7(2), there is a subgroup $D_{a,u}$ of $X_a \cap K_u$ isomorphic to this complement. By Lemmas 2–4, the group $D_{u,a}$ acts fixed point freely on $U_{a,u}/Z_{a,u}$.

Step 1: *We have* $U_u = U_{a,u}$.

The group G_a/K_a is isomorphic to a subgroup of $A := \mathrm{Aut}(\Gamma_a)/(\mathrm{Aut}(\Gamma_a))^\infty$ by Lemma 5 (G)(U) and 6(1). By Lemma 1, the group A is a 2-group of order at most 4 for each of our possibilities for Γ_a. Furthermore, if $p=2$, A is a subgroup of the cyclic group of field automorphisms. The group $K_u/K_u \cap X_a \simeq K_uX_a/X_a$ is a subgroup of G_a/X_a, which is isomorphic to a subgroup of A. In particular, if p is odd, that is if $p=3$ and $\Gamma_a \simeq Q(4,3)$ or $Q(4,9)$, we have $O_3(K_u) = O_3(K_u \cap X_a)$. If $p=2$, $\Gamma_a \simeq W(4)$, $Q^-(5,4)$ or $H(3,2^2)$ and $|A| = 2$, 4 or 2, respectively. In any case, $X_a \cap G_u$ induces $L_2(4)$ on the set $\Omega := \mathcal{P}(u)\backslash\{a\}$ of 5 points. We may verify that there is a field automorphism σ of order $|A|$ which fixes u and induces a transposition on 5 points on Ω. Thus any element in the coset $(X_a \cap G_u)\sigma$ induces an odd permutation on Ω, and therefore $(X_a \cap G_u)\sigma \cap K_u = \emptyset$. Thus $K_u = K_u \cap X_a$ if $\Gamma_a \simeq W(4)$ or $H(3,2^2)$. If $\Gamma_a \simeq Q^-(5,4)$, σ^2 fixes any element of Ω, but it does not centralize any element of order 15 of K_u (see Lemma 3(1)). Thus, for $p=2$, we always have $O_2(K_u) = O_2(K_u \cap X_a)$.

Step 2: *There is a normal subgroup* X_u *of* G_u *such that* $[X_u, D_{a,u}] \subseteq U_u$ *and* $X_u/K_u \simeq A_5$, M_{11}, A_6, A_6 *or* A_6 *if* $\Gamma_a \simeq Q(4,3)$, $Q(4,9)$, $Q^-(5,4)$, $H(4,2^2)$ *or* $W(4)$, *respectively.*

We denote by $\bar{T}$ the image of a subset T of G_u/U_u. Since $U_u = U_{a,u}$ and $K_u/(K_u \cap X_a)$ is isomorphic to a subgroup of $\mathrm{Aut}(\Gamma_a)/(\mathrm{Aut}(\Gamma_a))^\infty$, the group $\overline{D_{a,u}} = D_{a,u}U_u/U_u$ coincides with the normal subgroup $O_{p'}(\overline{K_u})$ of $\overline{G_u}$. We set $X_u/U_u := C_{\overline{A_u}}(\overline{D_{a,u}})$. Then $\overline{X_u}$ is a normal subgroup of $\overline{A_u}$ and $\overline{A_u}/\overline{X_u}$ ($\subseteq \mathrm{Aut}(\overline{D_{a,u}})$) is solvable, as $\overline{D_{a,u}}$ is cyclic. Since A_u/K_u is a non-abelian simple group, we have $\overline{A_u} = \overline{X_uK_u}$, and therefore $X_u/K_u \cap K_u \simeq \overline{X_u}/\overline{K_u} \cap \overline{K_u} \simeq \overline{A_u}/\overline{K_u}$. The claim follows.

Step 3: *The group* $Z_{a,u}$ *is normal in* G_u.

If $\Gamma_a \simeq Q(4,3)$, $Q(4,9)$ or $Q^-(5,4)$, we have $K_a = 1$ by Lemmas 2, 3 and 7(3). In this case, the group $Z_{a,u} = Z(U_{a,u})$ is normal in G_u by Step 1. If $\Gamma_a \simeq H(4,2^2)$, $|K_a| = 1$ or 2 by Lemma 7(2). If $K_a = 1$, $Z_{a,u} = Z(U_{a,u})$ is normal in G_u by Step 1. If $|K_a| = 2$, the group $M := \langle K_b \mid b \in \mathcal{P}(u)\rangle$ is a G_u-invariant group of order at most $2^{|\mathcal{P}(u)|} = 2^6$. If $M = K_a$, then $K_a = K_b = K_a \cap K_b = 1$ by Lemma 7(2), which is a contradiction. Thus M/K_a is a $X_a \cap G_u$-invariant subgroup of U_u/K_a. Since $Z_{a,u}/K_a$ is the unique $X_a \cap G_u$-invariant proper non-trivial subgroup of U_u/K_a by Lemma 4, we have $M = Z_{a,u}$.

Assume that $\Gamma_a \simeq W(4)$. As remarked before Step 1, in this case the group $Z_{a,u}/K_a$ coincides with the group of symmetries about the isotropic line u of Γ_a. For each point $b \in \mathcal{P}(u)$, there is a subgroup $M_{a,u}/K_a$ of $(X_a \cap K_u)/K_a$ isomorphic to $L_2(4)$ (see Lemma 2(E.x)). Since K_a is an elementary abelian group of order at most 4 by Lemmas 2 and 7(1)(2), $M_{a,u} = C_{M_{a,u}}(K_a)K_a$. We fix $M_{b,u}$ for any $b \in \mathcal{P}(u)$, and denote its commutator subgroup by $L_{b,u}$. The group $L_{a,u}$ is perfect, and so $L_{a,u}$ commutes with K_a. Moreover, $L_{a,u}K_a/K_a \simeq L_2(4)$. By Lemma 2(E.x), we have $[L_{a,u}, Z_{a,u}] \subseteq K_a$. Since $L_{a,u}$ is perfect and commutes with the normal subgroup K_a of $Z_{a,u}$, we have $[L_{a,u}, Z_{a,u}] = 1$ by the three subgroup lemma. Since $A_6 \simeq A_u/K_u = L_{a,u}L_{b,u}K_u/K_u$ for a point b distinct from a, we have $(L_{a,u} \cap L_{b,u})K_u/K_u \simeq A_4$. In particular, $L_{a,u} \cap L_{b,u}$ contains an element of order 3. We may verify in $S_4(4)$ that any element of order 3 in $L_{a,u}/K_a$ acts fixed point freely on $U_u/Z_{a,u}$. Now, suppose $Z_{a,u} \neq Z_{b,u}$ for a point b distinct from a. Then t does not centralize the non-trivial subgroup $Z_{a,u}Z_{b,u}/Z_{a,u}$, which contradicts the fact that $[L_{b,u}, Z_{b,u}] = 1$. Thus $Z_{a,u} = Z_{b,u}$ for all points b on u, and the group $Z_{a,u}$ is normal in G_u.

Final step. Since G_u/K_u acts on U_u/U_u', the groups X_u/U_u and $D_{a,u}U_u/U_u$ act on the 2-dimensional module $U_u/Z_{u,a}$ (see Step 2) over the field $\mathbf{F}_r$, where $r = 3$, 9, 4, 4 or 4 if

$\Gamma_a \simeq Q(4, 3)$, $Q(4, 9)$, $Q^-(5, 4)$, $H(4, 2^2)$ or $W(4)$, respectively. Since $D_{a,u}$ acts fixed point freely on $U_u/Z_{a,u}$ and $[X_u, D_{a,u}] \subseteq U_u$, X_u/U_u preserves an $\mathbf{F}_r$-structure afforded by the action of $D_{a,u}$, and therefore X_u/U_u induces a subgroup of $GL_2(r)$. Since X_u/U_u has a quotient group isomorphic to A_5, M_{11}, A_6, A_6 or A_6 for $\Gamma_a \simeq Q(4, 3)$, $Q(4, 9)$, $Q^-(5, 4)$, $H(4, 2^2)$ or $W(4)$, we conclude that G_u/U_u acts trivially on $U_u/Z_{a,u}$. However, X_u/U_u involves the group $L_{a,u}$ acting non-trivially on $U_u/Z_{a,u}$, which is a contradiction. $\square$

Summarizing the results in this section, we have the following:

PROPOSITION 13. *Let Γ be a flag-transitive classical $c.C_2$-geometry Γ of order (s, t) with $s \geq 3$, admitting a flag-transitive group G. Then the isomorphism class of $(\Gamma_a, (G_a/K_a)')$ is one of the following five possibilities:*

$$(W(3), S_4(3)), \quad (Q^-(5, 3), O_6^-(3)), \quad (H(3, 2^2), U_4(2)),$$
$$(H(3, 3^2), U_4(3)) \quad \textit{or} \quad (H(3, 3^2), L_3(4)).$$

The latter two cases are treated in [29], by the same method as in this paper. Thus, we will consider the other three cases in the remainder of this paper.

4. THE CASE $\Gamma_a \simeq W(3)$

In this case, we have $G_a/K_a \simeq S_4(3)$ or $S_4(3).2$, and $K_a \simeq 1$ or 3 by Lemmas 7(1) and 2(O.1). Suppose $K_a = 1$. Then $X_a \cap G_u \simeq 3^3{:}S_4$, $D_u := O_3(K_u) \simeq 3^3$, and $G_u/K_u \simeq S_5$. Since $GL_3(3)$ is a $5'$-group, the kernel of the action of G_u on D_u is of index at most 4. However, there is a subgroup $X_a \cap G_u$ isomorphic to S_4 acting faithfully on D_u. Thus $K_a \simeq 3$. Let k be a generator of K_a. Since the Schur multiplier of $X_a/K_a \simeq S_4(3)$ is of order 2, the extension X_a/K_a splits, and we have $X_a = Y_a \times K_a$, with $Y_a := X_a' \simeq S_4(3)$.

We take generators e_0, e_1, e_2, e_3, e_4, e_5 of Y_a satisfying the following relations (1), the Steinberg relations, where we set $r := e_0e_4e_0$, $h := r^2$ and $s := e_1e_5e_1$.

$$\begin{array}{llll}
e_i^3 = 1 & \text{for any } i = 0, \ldots, 5,\ r^4 = 1,\ s^2 = 1,\ (rs)^4 = 1; & & \\
e_1^s = e_5, \quad e_2^s = e_4, & e_3^s = e_3^{-1}, \qquad e_0^r = e_4, & e_1^r = e_3, & e_2^r = e_2; \\
[e_i, h] = 1 & \text{for } i = 0, 2, 4;\ e_i^h = e_i^{-1} \ \text{ for } i = 1, 3, 5; & & \\
[e_i, e_j] = 1 & \text{for any } 0 \leq i, j \leq 5 \text{ with } |i - j| \leq 1; & & \\
[e_0, e_2] = 1 & [e_0, e_3] = e_1e_2, \qquad [e_1, e_3] = e_2^{-1}, & [e_1, e_4] = e_2^{-1}e_3; & \\
[e_2, e_4] = 1; & [e_2, e_5] = e_3e_4, \qquad [e_3, e_5] = e_4^{-1}. & &
\end{array} \tag{1}$$

Note that the above relations (1) give a presentation for $S_4(3)$. For convenience, we also give an explicit matrix representation for each of the elements above. We identify $S_4(3)$ with the quotient group of the matrix group $\{X \in GL_4(3) \mid XJ_4^tX = J_4\}$ by its center $\langle -I \rangle$, where J_4 is the 4×4 matrix with (i, j) entry -1 if $(i, j) = (4, 1)$, $(3, 2)$, 1 if $(i, j) = (2, 3)$, $(4, 1)$ and 0 otherwise. Then the elements $e_0, \ldots, e_5$ above correspond to the image of the following matrices:

$$e_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

$$\mathrm{r}=\begin{pmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\quad \mathrm{h}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},\quad \mathrm{s}=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}.$$

Since $[k, Y_a]=1$, we have the relations:

$$k^3=1, \quad [k, e_i]=1 \qquad \text{for any } i=0, \ldots, 5. \tag{3}$$

If $G_a/K_a \simeq S_4(3).2$, there is an involution d of G_a satisfying the following relations:

$$e_i^d=e_i^{-1} \quad \text{for } i=0, 2, 3, 4; \qquad e_j^d=e_j \quad \text{for } j=1, 5. \tag{4}$$

The outer automorphism of $S_4(3)$ corresponding to d is given by $X \to KXK$ for $X \in S_4(3)$, under the above identification (2), where K is the 4×4 diagonal matrix with (i, i) entries $1, 1, -1, -1$ for $i=1, 2, 3, 4$, respectively.

We set $E:=\langle e_1, e_2, e_3\rangle\langle e_0, e_4\rangle$ and $F:=\langle e_2, e_3, e_4\rangle\langle e_1, h, e_5\rangle$. Then $E\simeq 3_+^{1+2}{:}\,2A_4$ and $F\simeq 3^3{:}\,S_4$. They are two maximal parabolic subgroups of Y_a containing a Borel subgroup $E\cap F=\langle e_2, e_3, e_4\rangle\langle e_1, h\rangle\simeq 3^3{:}\,S_3$. Then the groups E and F correspond to the stabilizers of an isotropic point and an isotropic line of the $S_4(3)$-quadrangle $W(3)$, respectively. Thus, there is a flag $(a, \{a, b\}, u)$ with $E=Y_a\cap G_{a,b}$ and $F=Y_a\cap G_u$. We have $K_u=\langle k, e_2, e_3, e_4\rangle$ if $G_a=X_a$. If $G_a=X_a\langle d\rangle$, we have $K_u=\langle k, e_2, e_3, e_4, d\rangle$, since F induces the full automorphism S_4 on $\mathscr{P}(u)$. We denote $D_u:=O_3(K_u)=\langle k, e_2, e_3, e_4\rangle\simeq 3^4$.

Since G_u/K_u is a transitive extension of S_4, we have $G_u/K_u\simeq S_5$. Then $G_u/D_u\simeq S_5$ if $G_a=X_a$, and $G_u/D_u\simeq 2S_5$ if $G_a=X_a\langle d\rangle$. For a while, we will consider the latter case. In this case, the image $\bar{d}$ of d in $\overline{G_u}=G_u/D_u$ lies in the center of $\overline{G_u}$. Since D_u is a 3-group, we have $\overline{G_u}=C_{\overline{G_u}}(\bar{d})=C_{G_u}(d)D_u/D_u$. As $C_{D_u}(d)=1$ by (4), $S:=C_{G_u}(d)\simeq 2S_5$. Since S contains a subgroup $\langle d\rangle\times\langle e_1, e_5, h\rangle$ of index 5, the extension $S/\langle d\rangle$ splits. The group $S\simeq S_5\times 2$ contains two subgroups isomorphic to S_5, since $S/S'\simeq E_4$ and $S'\simeq A_5$. One of them, say S_1, contains the subgroup $\langle e_1, e_5, h\rangle$ isomorphic to S_4, and the other contains $\langle e_1, e_5, hd\rangle\simeq S_4$.

Now we define $X_u:=G_u$ if $G_a=X_a$ and $X_u:=D_u{:}\,S_1$ if $G_a=X_a\langle d\rangle$, where S_1 is the subgroup above. We also let $X:=\langle X_a, X_u\rangle$. If $G_a=X_a$, then $G=X$. If $G_a=X_a\langle d\rangle$, $G=\langle G_a, G_{\{a,b\}}, G_u\rangle=\langle G_a, G_u\rangle=X\langle d\rangle$, as $G_u=X_a\langle d\rangle$. In the remainder of this section, we will try to find a set of relations for generators of X.

The group X_a is generated by the elements $e_0, \ldots, e_5, k$ satisfying the relations (1) and (3). We have $X_u/D_u\simeq S_5$ and $S_4\simeq\langle e_1, e_5, h\rangle\subseteq X_u$ by the definition of X_u. We let $h_1:=h^{e_1}=e_1h$ and $h_2:=h^{e_5}=e_5h$; then by (1) three involutions h_1, h and h_2 satisfy the standard presentation for S_4:

$$h_1^2=h^2=h_2^2=1, \qquad (h_1h)^3=(hh_2)^3=1, \qquad (h_1h_2)^2=1. \tag{5}$$

Since $X_u/D_u\simeq S_5$, there is an element v of X_u such that

$$v^2\equiv 1, \quad [h_1, v]\equiv 1, \quad [h, v]\equiv 1 \quad\text{and}\quad (h_2v)^3\equiv 1 \qquad (\operatorname{mod} D_u). \tag{6}$$

The element v acts on $D_u\langle h, e_1\rangle$ ($\subseteq G_a\cap G_b$) by (6) and the points a and b are the only points on u fixed by $D_u\langle h, e_1\rangle$. Thus v acts on $\{a, b\}$, and so $a^v=b$, as $v\notin G_a$. Furthermore, as D_u is a 3-group, we may choose $v^2=1$:

$$v^2=1, \qquad a^v=b. \tag{7}$$

We will consider the action of v on $D_u\simeq 3^4$. With respect to the basis (k, e_2, e_3, e_4)

of D_u, we have the following actions of h_1, h and h_2 by (1):

$$h_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&-1&0\\0&1&1&1\end{pmatrix},\quad h=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},\quad h_2=\begin{pmatrix}1&0&0&0\\0&1&-1&1\\0&0&-1&-1\\0&0&0&1\end{pmatrix}. \tag{8}$$

By (6), v acts on $C_{D_u}(h_1)=\langle k, e_2\rangle$. Since v exchanges K_a and K_b ($\neq K_a$) by (7), $k^v=k^\alpha e_2^\beta$ for some $\alpha=0$, ±1 and $\beta=\pm1$. By replacing k by k^{-1} if necessary, we may assume that the relations (1)–(8) and the following relation (9) hold:

$$k^v=k^\alpha e_2 \qquad \text{for some } \alpha=0,\ \pm1. \tag{9}$$

Since $v^2=1$, we have $e_2^v=(k^{-\alpha}k^v)v=k^{(1-\alpha^2)}e_2^{-\alpha}$. Since v acts on $[D_u, h]=\langle e_3\rangle$ by (6), the action of v on D_u is of the following form for suitable $\alpha=0$, ±1, $\beta=\pm1$, γ, $\delta=0$, ±1 and $\eta=\pm1$:

$$u=\begin{pmatrix}\alpha&1&0&0\\1-\alpha^2&-\alpha&0&0\\0&0&\beta&0\\\gamma&\delta&0&\eta\end{pmatrix}. \tag{10}$$

By the condition $[h_1, v]\equiv1 \pmod{D_u}$, we have $\alpha=\pm1$, $\beta=\eta=-\alpha$. As $v^2=1$, we then have $\gamma=-\alpha\delta$. By the condition $(h_2v)^3\equiv1 \pmod{D_u}$, a direct computation shows that $\alpha=-1$, $\beta=\gamma=\delta=\eta=1$. Thus,

$$k^v=k^{-1}e_2, \qquad e_2^v=e_v, \qquad e_3^v=e_3 \qquad \text{and} \qquad e_4^v=ke_2e_4. \tag{11}$$

In particular, $C_{D_u}(v)=\langle e_2, e_3\rangle$ and $\{x\in D_u \mid x^v=x^{-1}\}=1$. Since h_1, h, h_2 and v are involutions, it follows from (6) that the elements $[h, v]$, $[h_1, v]$ and $(h_2v)^3$ are elements of D_u inverted by v. Thus we have:

$$[h, v]=1, \qquad [h_1, v]=1 \qquad \text{and} \qquad (h_2v)^3=1. \tag{12}$$

Finally, we will consider e_0^v. By (7) and (12), v acts on $C:=C_{G_a\cap G_b}(h)$. If $G_a=X_a$, $C=\langle k, e_2\rangle\times\langle e_0, e_4\rangle$. If $G_a=X_a\langle d\rangle$, $C=(\langle k, e_2\rangle\times\langle e_0, e_4\rangle)\langle d\rangle$ by (4). In this case, $C'=\langle k, e_2\rangle\times\langle e_0, e_4\rangle=X_a\cap C$ by (4), since $[d, S_1]=1$ and S_1 acts transitively on $\{K_b \mid b\in\mathcal{P}(u)\}$, where S_1 is the subgroup of X_u isomorphic to S_5 as before. Thus, in any case, v acts on $X_a\cap C$ and $Z(X_a\cap C)=\langle k, e_2\rangle\times\langle h\rangle$. Therefore v acts on the group $\overline{X_a\cap C}=(X_a\cap C)/Z(X_a\cap C)$ ($\simeq A_4$), in which there are three involutions $\bar r$, $(\overline{e_0})^{-1}\overline{e_4}$ and $\overline{e_4}(\overline{e_0})^{-1}$. Then v fixes $\bar r$ or one of $(\overline{e_0})^{-1}\overline{e_4}$ and $\overline{e_4}(\overline{e_0})^{-1}$. However, in the latter case, v also fixes $\bar r=\overline{e_0e_4e_0}$, since v fixes $\overline{e_4}$ by (11). Thus, in any case, we have $r^v=r(k^ae_2^bh^c)$ for some a, b, c. Since k and e_2 are 3-elements centralizing $\langle e_0, e_4\rangle$, we have $r^v=r$ or $r^v=rh=r^3$. As $e_4^r=e_0$ by (1), we have $e_0^v=e_4^{rv}=e_4^{vr^i}=(ke_2e_4)^{r^i}=ke_2e_0$ for $i=1, 3$ by (1) and (4). Thus we have the final relation:

$$e_0^v=ke_2e_0. \tag{13}$$

Summarizing, we have proved the following statement:

PROPOSITION 14. *Assume that $\Gamma_a\simeq W(3)$. Then there exists a normal subgroup X of G of index at most 2 with generators $e_0, \ldots, e_5$, k and v satisfying the following set of* *relations, where we set* $r := e_0e_4e_0$, $h := r^2$, *and* $s := e_1e_5e_1$:

$$e_i^3 = 1 \quad \text{for any } i = 0, \ldots, 5, \qquad r^4 = 1, \qquad s^2 = 1, \qquad (rs)^4 = 1;$$
$$e_1^s = e_5, \quad e_2^s = e_4, \quad e_3^s = e_3^{-1}; \qquad e_0^r = e_4, \quad e_1^r = e_3, \quad e_2^r = e_2;$$
$$[e_i, h] = 1 \quad \text{for } i = 0, 2, 4, \qquad e_i^h = e_i^{-1} \quad \text{for } i = 1, 3, 5;$$
$$[e_i, e_j] = 1 \qquad \text{for any } 0 \leqslant i, j \leqslant 5 \text{ with } |i - j| \leqslant 1;$$
$$[e_0, e_2] = 1, \qquad [e_0, e_3] = e_1e_2, \qquad [e_1, e_3] = e_2^{-1}, \qquad [e_1, e_4] = e_2^{-1}e_3;$$
$$[e_2, e_4] = 1, \qquad [e_2, e_5] = e_3e_4, \qquad [e_3, e_5] = e_4^{-1};$$
$$k^3 = 1, \quad [k, e_i] = 1 \quad \text{for all } i = 0, \ldots, 5; \qquad v^2 = 1, \quad k^v = k^{-1}e_2;$$
$$e_0^v = ke_2e_0, \qquad [e_1, v] = [e_2, v] = [e_3, v] = 1, \qquad e_4^v = ke_2e_4, \qquad (e_5hv)^3 = 1.$$

Now we will prove theorem (a) in Section 1 (see also the description of the method in Section 1). Assume that Γ is an arbitrary flag-transitive $c.C_2$-geometry with point residues $\Gamma_a \simeq W(3)$. Then there is a normal subgroup X of a flag-transitive group G of index at most 2 satisfying the conditions in Proposition 14 above. Let $\tilde{X}$ be a group with a presentation the relations in Proposition 14, and denote by $\tilde{X}_x$ the subgroups generated by elements corresponding to the generators of X_x $(x = a, r, u)$. By Propositoin 14, there is an epimorphism η from $\tilde{X}$ onto X. The epimorphism η induces the isomorphisms $\tilde{X}_x \simeq X_x$ for $x = a, r, u$, since there is no non-trivial element of X fixing all points (lines or planes). Thus there is a 2-covering from the coset geometry $\tilde{\Gamma}$ defined by $\tilde{X}$ and $\tilde{X}_x$ $(x = a, r, u)$ onto Γ, since we may verify that X is flag-transitive on Γ. In particular, $\tilde{\Gamma}$ is a flag-transitive classical $c.C_2$-geometry with point residues $W(3)$. Note that there is a flag-transitive classical $c.C_2$-geometry Γ_0 on 176 points with point residues $W(3)$, admitting a simple group $U_5(2)$. Now, by coset enumeration using Cayley, we have $|\tilde{X} : \tilde{X}_a| = 176$. Thus $\tilde{\Gamma}$ is isomorphic to Γ_0 and $\tilde{X} \simeq U_5(2)$. Since Γ is a quotient of Γ_0 and $U_5(2)$ is simple, we have $\Gamma_0 \simeq \Gamma$. Thus, we have proved the following:

THEOREM 15. *Assume that* Γ *is a flag-transitive classical* $c.C_2$*-geometry with* $\Gamma_a \simeq W(3)$. *Then* Γ *is isomorphic to the simply connected geometry on* 176 *points admitting* $U_5(2)$.

5. THE CASE $\Gamma_a \simeq Q^-(5, 3)$

In this case, $K_a = 1$ by Lemmas 7(3) and 3(u). Then $X_a \simeq U_4(3)$, and therefore there are generators e_0, e_3, e_4, s, f of X_a satisfying the Steinberg relations (1), where we set $r := e_0e_4e_0$, $h := r^2$, $e_5 := e_1^s$, $e_1 := (e_3^{-1})^r$, $e_2 := [e_1, e_3]^{-1}$, $f_i := e_i^f$ for $i = 1, 3$ (see [29, Lemma 6]):

$$\begin{aligned}
&(rs)^4 = 1, \qquad e_0^3 = e_1^3 = 1, \qquad h^2 = 1, \qquad h = f^2, \qquad s = e_1e_5e_1,\\
&e_4 = e_0^r, \quad e_2^s = e_4, \quad s^2 = 1; \qquad e_3^s = e_3^{-1}, \quad e_2^r = e_2, \quad [e_0, h] = 1;\\
&[e_0, e_1] = 1, \quad [e_0, e_3] = e_1e_2, \quad [e_1, f_3] = 1; \qquad e_0^f = e_0^{-1}, \quad [f_1, e_1] = 1, \quad e_1^h = e_1^{-1};\\
&f^s = f^{-1}, \quad f^r = f^{-1}; \qquad (fe_1e_5)^5 = 1.
\end{aligned} \tag{1}$$

Note that these relations are also satisfied if we replace f by f^{-1}. For convenience, we will give explicit matrices corresponding to these elements. We identify $SU_4(3)$ with the group of matrices $\{X \in SL_4(9) \mid XJ_4^t\bar{X} = J_4\}$, where J_4 is the 4×4 matrix with (i, j)th entry 1 if $i + j = 5$ and 0 otherwise. The group $U_4(3)$ is the quotient group of $SU_4(3)$ by its center $\langle \eta I \rangle$, where $\eta := \zeta^2$ and ζ is a generator of $\mathbf{F}_9^*$. Then, modulo

$\langle \eta I\rangle$, the above elements correspond to the following matrices:

$$e_0=\begin{pmatrix}1&0&0&0\\0&1&\eta&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad e_1=\begin{pmatrix}1&0&0&0\\-1&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix},\quad e_2=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\\eta&0&0&1\end{pmatrix},$$
$$e_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\\eta&0&1&0\\0&\eta&0&1\end{pmatrix},\quad e_4=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&\eta&1&0\\0&0&0&1\end{pmatrix},\quad e_5=\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&-1\\0&0&0&1\end{pmatrix},$$
$$r=\begin{pmatrix}1&0&0&0\\0&0&\eta&0\\0&\eta&0&0\\0&0&0&1\end{pmatrix},\quad s=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}, \tag{2}$$
$$f=\begin{pmatrix}\zeta&0&0&0\\0&\zeta^{-1}&0&0\\0&0&-\zeta^{-1}&0\\0&0&0&-\zeta\end{pmatrix},\quad h=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}.$$

We have $e_i^f=e_i^{-1}$ for $i=0, 2, 4$. The elements $e_0, \dots, e_5$ satisfy the Steinberg relations for $S_4(3)$ (see Section 4(1)), and therefore they generate a subgroup of $U_4(3)$ isomorphic to $U_4(2)\simeq S_4(3)$.

We set $E:=\langle e_2, e_3, f_3, e_4\rangle\langle e_1, e_5, f\rangle$ and $F:=\langle e_1, f_1, e_2, e_3, f_3\rangle\langle e_0, e_4, f\rangle$. Then $E\simeq 3^6{:}A_6$ and $F\simeq 3^{1+4}{:}2S_4$ are two maximal parabolic subgroups of $X_a\simeq U_4(3)$ containing a Borel subgroup $E\cap F=\langle e_2, e_3, f_3, e_4\rangle\langle e_1, f_1, f\rangle\simeq 3^4{:}(3^2.4)$. Then there is a flag $(a, \{a, b\}, u)$ such that $E=X_a\cap G_b$ and $F=X_a\cap G_u$. We have $F/X_a\cap K_u\simeq S_4$ and so $G_u/K_u\simeq S_5$.

We will construct a normal subgroup X_u of G_u containing D_u, e_0, e_4 and f with $X_u/D_u\simeq S_5$. Let $Y_u:=O^2(G_u)$; that is, the subgroup of G_u generated by all the elements of odd order. Then Y_u contains D_u and $\langle e_0, e_4\rangle\simeq 2A_4$. Since $G_u/K_u\simeq S_5$, we have $Y_u/Y_u\cap K_u\simeq A_5$. We will show that $Y_u\cap K_u$ $(\supseteq D_u\langle h\rangle)$ centralizes Y_u modulo $D_u\langle h\rangle=K_u\cap X_a$.

The group $K_u/K_u\cap X_a$ is a subgroup of G_a/X_a, which is isomorphic to a subgroup of D_8 (see Lemma 1). Then any element of order 5 of G_u centralizes $K_u/D_u\langle h\rangle$. Furthermore, $[\langle e_0, e_4, f\rangle, K_u]\subseteq X_a\cap K_u$, since $\langle e_0, e_4, f\rangle\subseteq X_a\trianglelefteq G_a\supseteq K_u$. Since Y_u is generated by D_u together with the elements of order 3 conjugate to e_0 and the elements of order 5, we have shown that $[Y_u, K_u]\subseteq D_u\langle h\rangle$. In particular, the group Y_u/D_u is a central extension of $A_5\simeq Y_u/Y_u\cap K_u$ by $Y_u\cap K_u/D_u$. Since the Schur multiplier of A_5 is of order 2 and Y_u contains $2A_4$, we have $Y_u/D_u=A_u\times T_u$, where $A_u\simeq 2A_5$ and T is a 2-group. Since $O^2(Y_u/D_u)=Y_u/D_u$, we have $Y_u/D_u\simeq 2A_5$. Then, if we let $X_u:=Y_u\langle f\rangle$, we have $X_uK_u=G_u$ and X_u is normalized by K_u from the above computation. Thus the group X_u is a normal subgroup of G_u with the desired properties. Furthermore, since any coset of G_a/X_a contains an element stabilizing u, the group $X:=\langle X_a, X_u\rangle$ is normalized by $G=\langle G_a, G_u\rangle$.

We will find generators and relations for the normal subgroup X of G. The element h corresponds to the central involution of $X_u/D_u\simeq 2S_5$ and acts fixed point freely on $D_u/Z(D_u)$. Thus $C_{X_u}(h)$ is isomorphic to $(3\times 2.A_5).2$. Since it contains $(Z(D_u)\times\langle e_0, e_4\rangle)\langle f\rangle$, the group $S:=(C_{X_u}(h))^\infty\langle f\rangle$ is a group isomorphic to $2S_5$ containing e_0, e_4, f. We set $f_1:=f^{e_4}=e_4f$ and $f_2:=f^{e_0}=e_0f$. Then f_1, f^{-1}, f_2 satisfy the standard relations for $2S_4$: $(f_i)^2 = h = f^{-2}$ $(i = 1, 2)$, $(f_1 f^{-1})^3 = (f^{-1} f_2)^3 = 1$ and $[f_1, f_2] = h$. Thus there is an element $v \in S$ such that

$$v^2 = h, \quad [f_1, v] = h, \quad [f, v] = h \quad \text{and} \quad (vf_2^{-1})^3 = 1. \tag{3}$$

Since $X_u = D_u S = \langle X_u \cap X_a, v \rangle$, we have $X = \langle X_a, v \rangle$. We will find relations between v and the generators e_0, e_3, e_4, s, f of X_a.

Since the element v is conjugate to $f \in X_u \backslash C_{X_u}(e_2)$, we have $e_2^v = e_2^{-1}$. Thus we have the following relations, by rewriting (3) in terms of e_4, e_0, f and v:

$$v^2 = h, \quad e_2^v = e_2^{-1}, \quad e_4^v = e_4, \quad f^v = f^{-1} \quad \text{and} \quad (vf^{-1}e_0^{-1})^3 = 1. \tag{4}$$

Since v normalizes $D_u\langle e_4, f\rangle (\subseteq G_a \cap G_b)$ and a, b are the only points on u fixed by this group, v interchanges a and b. Then v acts on $L := O_3(G_a \cap G_b)(G_a \cap G_b)^\infty = \langle e_2, e_3, f_3, e_4\rangle \langle e_1, e_5, f\rangle \simeq 3^4 \colon A_6$. By (4), v acts on $C := C_L(h) = \langle e_2, e_4\rangle\langle s, f\rangle$ and also on $\langle e_2, e_4\rangle$. Since $\langle s, f\rangle \simeq D_8$, we have $s^v \equiv sm \pmod{\langle e_2, e_4\rangle}$ for some $m = 1$, h, f or f^{-1}. Since $C_{\langle e_2, e_4\rangle}(s) = \langle e_2 e_4\rangle$, s^v centralizes $e_2^{-1}e_4$ by (4). Then we have $s^v = sf^i x$ for $i = 1, 3$ and $x \in \langle e_2, e_4\rangle$. Since s is an involution, we have $x^{sf^i} = x^{-1}$. We also have $f^i x f^{-i} x^v = 1$ from $s = s^h = s^{v^2}$. Then we can verify $x = 1$ using the actions of v, s on $\langle e_2, e_4\rangle$ (see (1) and (4)). Thus, we have:

$$s^v = sf \quad \text{or} \quad s^v = sf^{-1}. \tag{5}$$

By replacing f by $f^{-1} = fh$ and v by $v^{-1} = vh$, if necessary, we may assume that the relations (1), (4) and $s^v = sf$ hold. (Recall that the relations (1) hold if we replace f by f^{-1}.) Since $[e_4, v] = 1$, v acts on $C_{D_u}(e_4) = \langle e_3, f_3, e_2\rangle \simeq 3^3$. We may write $e_3^v = e_3^\alpha f_3^\beta e_2^\gamma$ for some $\alpha, \beta, \gamma = 0, \pm 1$. By computing both sides of the equation $e_3^{sv} = e_3^{vsf}$, we have $e_3^{-\alpha} f_3^{-\beta} e_2^{-\gamma} = f_3^{-\alpha} e_3^{-\beta} e_4^{-\gamma}$, since $e_3^s = e_3^{-1}$, $f_3^s = f_3$ and $e_2^s = e_4$. As $\langle e_2, e_3, f_3, e_4\rangle \simeq 3^4$, then we have $\alpha = \beta = \pm 1$ and $\gamma = 0$. Thus we conclude that there are generators e_0, e_3, e_4, s f, v of X satisfying the relations (1), (4) and

$$s^v = sf, \quad e_3^v = (e_3 f_3)^i, \quad \text{where } i = \pm 1, \tag{6.i}$$

Thus we have proved that there are generators e_0, e_3, e_4, s, f, v of X with one of the two possible sets of relations $\mathcal{R}_i$ $(i = \pm 1)$; that is, the union of the relations (1), (4) and (6.i). Assume that the relations $\mathcal{R}_{-1}$ hold. We set $e_0' := e_0^{-1}$, $e_3' := e_3^{-1}$, $e_4' := e_4^{-1}$, $s' := s$, $f' := f$ and $v' := v^{-1}$. The correspondence $e_i \mapsto e_i'$ $(i = 0, 3, 4)$, $s \mapsto s'$, $f \mapsto f'$ coincides with the composition of the field automorphism of $U_4(3)$ (see (2) above) and the automorphism of $U_4(3)$ fixing $\langle e_0, e_3, e_4, s\rangle$ and inverting f. Thus the Steinberg relations (1) holds if we replace e_i, s and f by e_i', s' and f', respectively $(i = 0, 3, 4)$. Since $(e_0 \cdot fv^{-1})^{-3} = 1$ by (4), we have $(v^{-1}f^{-1}e_0)^3 = (fv^{-1} \cdot e_0)^3 = 1$, and so the relations (4) also hold for e_i', f' and v'. Then we may easily verify that the elements e_i' $(i = 0, 3, 4)$, s', f' and v' satisfy the relations $\mathcal{R}_1$.

Summarizing the above argument, we have proved:

PROPOSITION 16. *Assume that $\Gamma_a \simeq Q^-(5, 3)$. Then there is a normal subgroup X of G of index at most 8 with generators e_0, e_3, e_4, f, s, v satisfying the following set of relations, where we set $r := e_0 e_4 e_0$, $h := r^2$, $e_5 := e_1^s$, $e_1 := (e_3^{-1})^r$, $e_2 := [e_1, e_3]^{-1}$, $f_j := e_j^f$ for $j = 1, 3$:*

$(rs)^4 = 1, \quad e_0^3 = e_1^3 = 1, \quad h^2 = 1, \quad h = f^2, \quad s = e_1 e_5 e_1,$

$e_4 = e_0^r, \quad s^2 = 1, \quad e_3^s = e_3^{-1}, \quad e_2^r = e_2, \quad e_4 = e_2^s, \quad [e_0, h] = 1;$

$[e_0, e_1] = 1, \quad [e_0, e_3] = e_1 e_2, \quad [e_1, f_3] = 1; \quad e_0^f = e_0^{-1}, \quad [f_1, e_1] = 1, \quad e_1^h = e_1^{-1};$

$f^s = f^{-1}, \quad f^r = f^{-1}; \quad (fe_1 e_5)^5 = 1;$

$v^2 = h, \quad f^v = f^{-1}, \quad s^v = sf, \quad (vf^{-1}e_0^{-1})^3 = 1,$

$e_3^v = e_3 f_3.$

Now let Γ be any $c.C_2$-geometry with $\Gamma_a \simeq Q^-(5,3)$, admitting a flag-transitive automorphism group G. By Proposition 16, there is a normal subgroup X of G with generators satisfying the set $\mathcal{R}$ of relations (1), (4) and (6.1). Let $\tilde{X}$ be the group with presentation $\mathcal{R}$, and let $\tilde{X}_x$ be the subgroups corresponding to the stabilizers of x in X $(x = a, r, u)$. As in the argument in the proof of Theorem 15, the coset geometries $\tilde{\Gamma}$ defined by $\tilde{X}$ and $\tilde{X}_x$ $(x = a, r, u)$ are simply connected, and there is a 2-covering from $\tilde{\Gamma}$ onto Γ. By coset enumeration using Cayley, we have $|\tilde{X}:\tilde{X}_a| = 275$, where $\tilde{X}_a = \langle e_0, e_3, e_4, s, f\rangle \simeq U_4(3)$. In [2] or [20], a flag-transitive classical $c.C_2$-geometry Γ_0 with point residues $Q^-(5,3)$ admitting $X_0 \simeq McL$ is constructed. Since X_0 is an epimorphic image of $\tilde{X}$, we have $\tilde{X} \simeq X_0$ and $\Gamma_0 \simeq \tilde{\Gamma} \simeq \Gamma$. Thus:

THEOREM 17. *Assume that Γ is a flag-transitive classical $c.C_2$-geometry with $\Gamma_a \simeq Q^-(5,3)$. Then Γ is isomorphic to the simply connected geometry on 275 points admitting McL.*

6. THE CASE $\Gamma_a \simeq H(3, 2^2)$

6.1. *General Arguments*

In this case, we have $K_a \simeq 1$ or 2 by Lemmas 7(1) and 3(1). Recall that X_a is the full inverse image of $(G_a/K_a)' \simeq U_4(2)$ in G_a. Since $X_a \cap G_u$ induces A_5 on the set $\mathcal{P}(u)\backslash\{a\}$, the transitive extension G_u/K_u induces A_6 or S_6 on $\mathcal{P}(u)$. We let X_u be the full inverse image of $(G_u/K_u)' \simeq A_6$ in G_u. We denote by X the subgroup of G generated by X_a and X_u: $X := \langle X_a, X_u\rangle$.

LEMMA 18. *The group X is a normal subgroup of G of index at most 2.*

PROOF. Let $(a, \{a,b\}, u)$ be a maximal flag. There is an element g of G_u exchanging two points a and b by the flag-transitivity of G, and so $G_{\{a,b\}} = (G_a \cap G_b)\langle g\rangle$. Then the residually-connectedness of Γ implies that $G = \langle G_a, G_{\{a,b\}}, G_u\rangle = \langle G_a, G_u\rangle$.

If $G_a = X_a$, $(G_u \cap G_a)/K_u \simeq A_5$ and $G_u/K_u \simeq A_6$. Thus $G_u = X_u$. In this case, we have $X = G$, and the claim holds trivially. If $X_a \neq G_a$, $G_a/K_a \simeq U_4(2).2$ and there is a 2-element of $G_a\backslash X_a$. Since there are 27 planes of $\mathcal{P}(a)$, on which X_a acts transitively, there is a 2-element t of $G_a\backslash X_a$ stabilizing u. By observing $G_a/K_a \simeq U_4(2).2$, we have $(G_a \cap G_u)/K_u \simeq S_5$ and $(G_a \cap X_u)/K_u \simeq A_5$. Then t induces a transposition on the set $\mathcal{P}(u)\backslash\{a\}$ of five points, and also on $\mathcal{P}(u)$. Thus $X_u\langle t\rangle = G_u$, as $G_u/K_u \simeq S_6$ and $X_u/K_u \simeq A_6$. In particular, the element t normalizes both X_a and X_u, and also $X = \langle X_a, X_u\rangle$. Furthermore, $G = \langle G_a, G_u\rangle = \langle X_a, X_u\rangle\langle t\rangle$. The claim follows. □

LEMMA 19. *The extension X_a/K_a splits.*

PROOF. Suppose X_a/K_a does not split. Then $|K_a| = 2$ and X_a is isomorphic to the unique perfect central extension of $U_4(2)$ by 2. Thus, $X_a \simeq \mathrm{Aut}(\Gamma)$, where Γ is the complex E_8-lattice [30, §2]. By [30 §2], K_u is an extra-special group of order 2^5 of negative type with center K_a. The group $M_u := \langle K_b \mid b \in \mathcal{P}(u)\rangle$ is a $X_a \cap G_u$-invariant elementary abelian 2-subgroup of K_u, by Lemma 7(2). Since $X_a \cap G_u$ act irreducibly on K_u/K_a, we have $M_u = K_a$. However, this implies that $K_a = K_b = 1$ by Lemma 7(2), which is a contradiction. □

In the remainder of this section, we will try to find some generators and relations of the group X. We set $Y_a := X_a'$ and let k be the generator of K_a if $|K_a| = 2$. By Lemma

17, we have $Y_a \simeq U_4(2)$. Then there are elements $e_0, \ldots, e_5$ and h of Y_a satisfying the Steinberg relations (1), where we set $r := e_0e_4e_0$, $s := e_1e_5e_1$ and $f_3 := e_3^h$:

$$\begin{aligned}
&e_i^2 = 1 \quad \text{for } i = 0, \ldots, 5, \quad h^3 = 1; \qquad [e_i, e_j] = 1 \quad \text{for any } 0 \le i, j \le 5 \text{ with } |i - j| \le 2,\\
&[e_i, e_{i+3}] = e_{i+1}e_{i+2} \quad \text{for } i = 0, 1, 2; \qquad [e_2, h] = 1, \quad [r, h] = 1, \quad h^s = h^{-1},\\
&(e_1h)^3 = 1, \qquad (he_1e_5)^5 = 1, \qquad [e_3, f_3] = 1, \qquad r^2 = s^2 = 1, \qquad (rs)^4 = 1,\\
&e_0^r = e_4, \qquad e_1^r = e_3, \qquad [e_2, r] = 1, \qquad e_1^s = e_5, \qquad e_2^s = e_4, \qquad [e_3, r] = 1.
\end{aligned} \tag{1}$$

If $K_a = 1$, we have $X_a = Y_a$, and therefore the relations (1) give a presentation for X_a. If $|K_a| = 2$, we have $X_a = K_a \times Y_a$ by Lemma 17, and therefore X_a is generated by $e_0, \ldots, e_5$, h and k satisfying the relations (1) and

$$k^2 = 1, \qquad [k, e_i] = 1 \quad \text{for } i = 0, \ldots, 5, \qquad [k, h] = 1. \tag{2}$$

For explicitness, we will identify the above elements with 4×4 unitary matrices of $\mathbf{F}_4$. We identify the group $SU_4(2) = U_4(2)$ with the group of 4×4 matrices X with entries in $\mathbf{F}_4$ satisfying $XJ_4{}^t\bar{X} = J_4$, where J_4 is the matrix with (i, j) entries 1 for $i + j = 5$ and 0 otherwise, and ${}^tX$ and $\bar{X}$ denote the transpose and the matrix obtained by squaring each entries of X, respectively. The above elements are given as follows, where ω is a cubic root of unity in $\mathbf{F}_4$:

$$\begin{aligned}
&e_0 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
e_1 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad
e_2 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 1 \end{pmatrix};\\
&e_3 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad
e_4 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
e_5 = \begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 \end{pmatrix};\\
&h = \begin{pmatrix} \omega & 0 & 0 & 0\\ 0 & \omega^2 & 0 & 0\\ 0 & 0 & \omega^2 & 0\\ 0 & 0 & 0 & \omega \end{pmatrix}, \quad
r = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
s = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{3}$$

The groups $E := \langle e_1, e_2, e_3 \rangle (\langle e_0, e_4 \rangle \langle h \rangle) \simeq 2_-^{1+4}{:}\,(S_3 \times 3)$ and $F := \langle e_2, e_3, f_3, e_4 \rangle \langle e_1, e_5, h \rangle \simeq 2^4{:}\,A_5$ are two maximal parabolic subgroups containing a Borel subgroup $E \cap F = \langle e_1, e_2, e_3, e_4, h \rangle \simeq 2^4{:}\,A_4$. Then there is a maximal flag $(a, \{a, b\}, u)$ such that $Y_a \cap G_b = E$ and $Y_a \cap G_u = F$. We have $K_u = \langle e_2, e_3, f_3, e_4 \rangle \simeq 2^4$ if $K_a = 1$ and $K_u = \langle k, e_2, e_3, f_3, e_4 \rangle \simeq 2^5$ if $|K_a| = 2$. The group X_u/K_u is isomorphic to A_6, which is a transitive extension of $(X_u \cap X_a)/K_u \simeq A_5$ on $\mathscr{P}(u) \setminus \{a\}$.

The actions of generators e_1, e_5, h of a Levi complement $(\simeq A_5)$ of $Y_a \cap X_u$ on $Y_a \cap K_u = \langle e_2, e_3, f_3, e_4 \rangle \simeq 2^4$ are as follows, where we represent them as matrices with respect to the basis (e_2, e_3, f_3, e_4):

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad
e_5 = \begin{pmatrix} 1 & 1 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
h = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

We will compute $C := C_{G_a \cap G_b}(h)$. If $G_a = X_a$, we have $G_a \cap G_b = X_a \cap X_b$. If $G_a = X_a \langle t \rangle$, we have $G_a \cap G_b = (X_a \cap X_b) \langle t \rangle$, where t corresponds to the field automorphism of Y_a; that is, $[e_j, t]=1$ for $j=0,\dots,4$ and $h^t=h^{-1}$ (see (3)). Thus, in any case, $C:=C_{G_a\cap G_b}(h)\subseteq X_a\cap X_b$, which is either E or $\langle k\rangle\times E$. Then we have $C=\langle e_2\rangle\times\langle e_0, e_4\rangle\times\langle h\rangle\simeq 2\times S_3\times 3$ if $K_a=1$ and $C=\langle k, e_2\rangle\times\langle e_0, e_4\rangle\times\langle h\rangle\simeq E_4\times S_3\times 3$ if $|K_a|=2$. Thus in any case, we have $C'=\langle e_0e_4\rangle\simeq 3$.

We will give an explicit identification of $\langle e_1, e_5, h\rangle$ as the alternating group A_5 on the six points $\mathcal{P}(u)$ fixing a. For simplicity, we number the six points on u as follows, where $m:=he_1e_5$ is an element of $X_u\cap X_a$ of order 5 (see (1)): $a_0:=a$, $a_{i+1}:=b^{m^i}$ $(i=0,1,\dots,4)$. Then, using (1), we may verify that e_1, e_5 and h induce the following permutations on $\{a_1,\dots,a_5\}$, where we simply write a_i as i:

$$e_1=(25)(34),\qquad e_5=(12)(34),\qquad h=(234). \tag{5}$$

The group X_u induces A_6 on $\mathcal{P}(u)$. Thus there is an element v of X_u corresponding to the permutation (01)(34). Note that X_u is generated by K_u, e_1, e_5, h and v. We have:

$$v^2\equiv 1,\quad h^v\equiv h^{-1},\ (e_5v)^3\equiv 1,\quad [e_1, v]\equiv 1\ (\operatorname{mod} K_u)\quad\text{and}\quad a^v=b. \tag{6}$$

Since h is an element of order prime to $|K_u|$, we may assume that:

$$h^v=h^{-1}. \tag{7}$$

In particular, the element v acts on $C:=C_{G_a\cap G_b}$, and so on $C'=\langle e_0e_4\rangle$ (see the remark above). Thus,

$$(e_0e_4)^v=(e_0e_4)^{\pm1}. \tag{8}$$

6.2. *The Case $K_u=1$*

Now we have to divide the cases $K_a=1$ or $|K_a|=2$. We first assume that $K_a=1$. Then $K_u=\langle e_2, e_3, f_3, e_4\rangle\simeq 2^4$. Using (6) and (4), we can determine the action of v on K_u by straightforward matrix calculations, where we take (e_2, e_3, f_3, e_4) as a basis of K_u:

$$v=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\1&0&0&1\end{pmatrix}. \tag{9}$$

By (4) and (9), we have $v^2\in C_{K_u}(v)\cap C_{K_u}(h)=\langle e_2\rangle$. Thus $v^2=1$ or e_2 by (3). Assume that we have $v^2=e_2$. We set $w:=ve_4$. As $e_4\in K_u$, the relations in (7) hold if we replace v by w. Furthermore, $h^w=h^{ve_4}=h^{-1}$ by (4), (8) and $w^2=v^2\cdot v^{-1}e_4v\cdot e_4=e_2e_2e_4e_4=1$ by (9). Thus, by taking w as a new v, if necessary, there is an element v of X_u with the following relations:

$$v^2=1,\quad h^v=h^{-1},\quad (e_5v)^3\in K_u,\quad [e_1, v]\in K_u\quad\text{and}\quad a^v=b. \tag{10}$$

As $e_4^v=e_2e_4$ by (9), we have $e_0^v=e_0e_2$ or re_2 if v centralizes or inverts e_0e_4, respectively (see (8)). Furthermore, by taking the conjugate of the equation $[e_0, e_3]=e_1e_2$ under v, we may verify that $e_1^v=e_1$ or $e_1e_2e_3$ if v centralizes or inverts e_0e_4, respectively.

We will show that v centralizes e_0e_4. Suppose v inverts e_0e_4. Then, by the remark above, we have:

$$ve_1v=(e_2e_3)e_1. \tag{11}$$

We will take an element $f\in X_u$ interchanging e_1 and v modulo K_u, and compute the conjugates of the both sides of (11) under f. For example, we take $f\in X_u$, inducing the permutation (05)(12) on $\mathscr{P}(u)$ (see the convention before (5)). Then f interchanges two permutations (01)(25) and (25)(34) induced by v and e_1, respectively. Thus we write $e_1^f = gv$ for some $g \in K_u$. Since e_1 is an involution, $g \in C_{K_u}(v)$. Similarly, we have $v^f = g'e_1$ for some $g' \in C_{K_u}(e_1)$, since v is an involution (see (10)). Now, by taking the conjugate of (11) under f, we have $(ve_1v)^f = x \cdot e_1^f$, where $x := (e_2e_3)^f$. Thus $g'e_1 \cdot gv \cdot g'e_1 = x \cdot gv$ and therefore $ve_1v = e_1x$, since $C_{K_u}(e_1) = C_{K_u}(v) = \langle e_2, e_3 \rangle$ (see (3), (6)). In particular, we have $x = e_2e_3$. However, we can show that $x = e_2$ as follows.

For two distinct points i and j on u ($0 \leqslant i, j \leqslant 5$), we denote by $K_{i,j}$ the subgroup of G fixing all the points collinear with i and j. Since $K_{i,j} \subseteq K_u$, we can verify that $K_{0,1} = \langle e_2 \rangle$ by observing X_a. Taking the conjugations of e_2 by m (see (4), (6)), we have $K_{0,1}^m = K_{0,2} = \langle e_2e_3e_4 \rangle$. Since v induces (01)(34) on $\mathscr{P}(u)$, we have $K_{1,2} = K_{0,2}^v = \langle e_3e_4 \rangle$ by (9). Then $K_{2,5} = K_{1,2}^{e_5e_1e_5} = \langle e_2e_3 \rangle$ by (5) and (3). Thus we have $x = (e_2e_3)^f = e_2$, since $K_{2,5}^f = K_{0,1}$. Hence we have a contradiction $e_2 = x = e_2e_3$, which implies that v centralizes e_0e_4. Summarizing, we have proved the following relations (see also (9)):

$$e_0^v = e_0e_2, \qquad e_i^v = e_i, \quad \text{for } i = 1, 2, 3 \qquad \text{and} \qquad e_4^v = e_2e_4. \tag{12}$$

Now only the relation between v and e_5 remains in the case $K_a = 1$. By (12), we have $e_5v = (sv)^{e_1}$, which is inverted by the involutions v and e_5. Furthermore, sv centralizes h by (1) and (7). Thus it follows from (9), (3), (8) that $(e_5v)^3 \in C_{K_u}(h)^{e_1} \cap C_{K_u}(e_5) \cap C_{K_u}(v) = 1$. Thus we have the final relation:

$$(e_5v)^3 = 1. \tag{13}$$

Summarizing the above arguments, we can conclude that:

PROPOSITION 20. *Assume that $\Gamma_a \simeq H(3, 2^2)$ and $K_a = 1$. Then there is a normal subgroup X of G of index at most 2 generated by elements e_0, e_1, e_2, e_3, e_4, e_5, h, v satisfying the following relations, where we set $r := e_0e_4e_0$, $s := e_1e_5e_1$ and $f_3 := e_3^h$:*

$$e_i^2 = 1 \quad \text{for } i = 0, \ldots, 5, \qquad h^3 = 1;$$
$$[e_i, e_j] = 1 \qquad \text{for any } 0 \leqslant i, j \leqslant 5 \text{ with } |i - j| \leqslant 2,$$
$$[e_i, e_{i+3}] = e_{i+1}e_{i+2} \quad \text{for } i = 0, 1, 2; \qquad [e_2, h] = 1, \quad [r, h] = 1, \quad h^s = h^{-1},$$
$$(e_1h)^3 = 1, \qquad (he_1e_5)^5 = 1, \qquad [e_3, f_3] = 1, \qquad r^2 = s^2 = 1, \qquad (rs)^4 = 1;$$
$$v^2 = 1, \qquad h^v = h^{-1}, \qquad e_0^v = e_0e_2, \qquad e_i^v = e_i \quad \text{for } i = 1, 2, 3,$$
$$e_4^v = e_2e_4, \qquad (e_5v)^3 = 1.$$

6.3. *The Case $|K_b| = 2$*

Now we will assume that $K_a \simeq 2$. All the relations up to (8) hold. The main difference from the case $K_a = 1$ lies in the possibilities of the action of v on K_u. In this case, $K_u = \langle k, e_2, e_3, f_3, e_4 \rangle \simeq 2^5$ and the subgroup X_a is generated by $e_0, \ldots, e_5, h$ and k with relations (1) and (2). Note that the actions of e_1, e_5 and h on K_u are obtained by (4) since they fix k. We also replace v by $w := ve_4$ if v inverts e_0e_4 (see (8)). The element w centralizes e_0e_4 and the relations (6), (7) are satisfied if we replace v by w, since $e_4 \in C_{K_u}(h)$. Thus, by replacing v by ve_4 if necessary, we conclude that the following relations hold:

$$h^v = h^{-1}, \qquad (e_0e_4)^v = e_0e_4, \qquad a^v = b,$$

and

$$v^2 \equiv 1, \qquad e_1^v \equiv e_1, \qquad (e_5v)^3 \equiv 1 \pmod{K_u}. \tag{14}$$

The element v interchanges the points a and b, and so $K_a = \langle k \rangle$ and K_b. Since the group K_bK_a/K_a is the group of symmetries about the isotropic point $\{a, b\}$ in Γ_a, we have $K_a \neq K_b \subseteq K_a\langle e_2 \rangle$. Thus:

$$K_b = \langle e_2 \rangle \qquad \text{or} \qquad K_b = \langle ke_2 \rangle. \tag{15}$$

LEMMA 21. *In the above, we have $K_b = \langle ke_2 \rangle$.*

PROOF. Assume that $K_b = \langle e_2 \rangle$. Then we have $k^v = e_2$ and $e_2^v = k$. Using (4) and (14), we can verify by direct computations that the action of v on K_u is as follows:

$$k^v = e_2, \qquad e_2^v = k, \qquad e_3^v = e_3, \qquad f_3^v = e_3 f_3, \qquad e_4^v = ke_2e_4. \tag{16}$$

In particular, $C_{K_u}(v) = \langle ke_2, e_3, e_2e_4 \rangle$. Then, by (4), we have $v^2 \in C_{K_u}(v) \cap C_{K_u}(h) \cap C_{K_u}(e_0e_4) = \langle ke_2 \rangle$. Thus $v^2 = 1$ or ke_2. If $v^2 = ke_2$, replace v by $w := ve_2$. Since $e_2 \in C_{K_u}(h) \cap C_{K_u}(e_0e_4)$, the relations in (14) and (16) are satisfied if we replace v by w. Furthermore, we have $w^2 = v^2 \cdot e_2^v \cdot e_2 = ke_2 \cdot k \cdot e_2 = 1$. Thus, we can conclude that there is an element v of X_u satisfying (16) and

$$v^2 \equiv 1, \qquad h^v = h^{-1}, \qquad (e_0e_4)^v = e_0e_4, \qquad a^v = b,$$

and

$$e_1^v \equiv e_1, \qquad (e_5v)^3 \equiv 1 \pmod{K_u}. \tag{17}$$

By (17) and (16), $e_0^v = e_0e_4e_4^v = ke_0e_2$. Then we can determine $e_1^v = [e_0^v, e_3^v]e_2^v$ (see the argument just after (10)). We have

$$e_0^v = ke_0e_2, \qquad e_1^v = ke_1e_2. \tag{18}$$

By (17) and (18), we have $r^v = (e_0e_4)^v e_0^v = r(ke_2)$ and so $f^v = (ke_2)v$. Since $[e_1, v] = ke_2$ by (18) and $[e_3, ke_2] = 1 = [e_3, v]$, we have $[e_3, v] = [e_3, (ke_2)v] = [e_3^v, r^v] = (ke_2)^v = ke_2$. However, this implies that $k = e_2 \in Y_a$, which is a contradiction. Thus the case $K_b = \langle e_2 \rangle$ does not occur. $\square$

By the above lemma, we have $K_b = \langle ke_2 \rangle$ and so $k^v = ke_2$ and $e_2^v = e_2$. Using (4) and (14), we can determine the action of v on K_u by arguments similar to those in the proof of the above lemma. We obtain:

$$k^v = ke_2, \qquad e_2^v = e_2, \qquad e_3^v = e_3, \quad f_3^v = e_3 f_3, \qquad e_4^v = e_2e_4. \tag{19}$$

Then we have $v^2 \in C_{K_u}(v) \cap C_{K_u}(h) \cap C_{K_u}(e_0e_4) = \langle e_2 \rangle$. If $v^2 = e_2$, we replace v by $w := vk$. Then there exists an element v of X_u satisfying (19) and (17). Furthermore, we have:

$$e_0^v = e_0e_2, \qquad e_1^v = e_1. \tag{20}$$

Now, by (20), we have $(e_5v)^3 = ((sv)^3)^{e_1} \in C_{K_u}(h)^{e_1} \cap C_{K_u}(e_5) \cap C_{K_u}(v) = \langle ke_3e_4 \rangle$. Suppose $(e_5v)^3 = ke_3e_4$. Then $(sv)^3 = (ke_3e_4)^{e_1} = ke_2e_4$. Since $s = e_1e_5e_1 = e_5e_1e_5$, we have $(sv)^{e_5} = e_1e_5ve_5$. As $ve_5ve_5ve_5 = ke_3e_4 =: y$, $e_5ve_5 = ve_5v \cdot y$ and so $(sv)^{e_5} = x \cdot y$ where $x := e_1ve_5v = (e_1e_5)^v$. Then $((sv)^3)^{e_5} = x^3 \cdot y \cdot y^x \cdot y^{x^2} = 1 \cdot y \cdot ke_3 \cdot ke_2e_4 = ke_2$, and therefore $(sv)^3 = (ke_2)^{e_5} = ke_2e_3e_4$, which contradicts the above result. Thus $(e_5v)^3 = 1$, and we obtain all the relations between v and $e_0, \ldots, e_5, h, k$.

PROPOSITION 22. *Assume that $\Gamma_a \simeq H(3, 2^2)$ and $|K_a| = 2$. Then there is a normal subgroup X of G of index at most 2 with generators $e_0, \ldots, e_5, h, k$ satisfying the* *following set of relations, where we set* $r := e_0e_4e_0$, $s := e_1e_5e_1$ *and* $f_3 := e_3^h$:

$$e_i^2 = 1 \quad \text{for } i = 0, \dots, 5, \qquad h^3 = 1;$$

$$[e_i, e_j] = 1 \qquad \text{for any } 0 \leqslant i, j \leqslant 5 \text{ with } |i - j| \leqslant 2,$$

$$[e_i, e_{i+3}] = e_{i+1}e_{i+2} \quad \text{for } i = 0, 1, 2; \qquad [e_2, h] = 1, \qquad [r, h] = 1, \qquad h^s = h^{-1},$$

$$(e_1h)^3 = 1, \qquad (he_1e_5)^5 = 1, \qquad [e_3, f_3] = 1, \qquad r^2 = s^2 = 1, \qquad (rs)^4 = 1;$$

$$e_0^r = e_4, \qquad e_1^r = e_3, \qquad [e_2, r] = 1, \qquad e_1^s = e_5, \qquad e_2^s = e_4, \qquad [e_3, r] = 1;$$

$$k^2 = 1, \qquad [k, e_i] = [k, h] = 1 \quad \text{for } i = 0, \dots, 5;$$

$$v^2 = 1, \qquad h^v = h^{-1}, \qquad k^v = ke_2, \qquad e_0^v = e_0e_2, \quad e_1^v = e_1,$$

$$e_2^v = e_2, \qquad e_3^v = e_2, \qquad e_4^v = e_2e_4, \qquad (e_5v)^3 = 1.$$

6.4. *Conclusion*

We will classify the flag-transitive $c.C_2$-geometries Γ with $\Gamma_a \simeq H(3, 3^2)$. Assume that Γ is any such geometry admitting a flag-transitive group G. By Propositions 22 and 20, we may observe that in the case $|K_a| = 2$ the elements e_0, e_3, e_4, s, f and v satisfy the relations stated for the case $K_a = 1$. Since $v^k = ve_2$ by (20) and $[k, X_a] = 1$, k normalizes the subgroup $Y := \langle e_0, e_3, e_4, s, f, v \rangle$: that is, Y is a normal subgroup of $X = Y\langle k \rangle$ of index at most 2 with generators satisfying the relations for the case $K_a = 1$. Thus, in each case, we can conclude that there is a normal subgroup X of G generated by e_0, e_3, e_4, s, f, v satisfying the relations in Proposition 20.

Let $\tilde{X}$ be a group with presentation the relations in Proposition 20, and denote by $\tilde{X}_x$ the subgroups generated by elements corresponding to the generators of X_x ($x = a, r, u$). We take the coset geometry $\tilde{\Gamma}$ defined by $\tilde{X}_x$ ($x = a, r, u$). As in an argument we have frequently used (see Section 1 and the proof of Theorem 15), there is an epimorphism from $\tilde{X}$ onto X inducing a 2-covering from $\tilde{\Gamma}$ onto Γ. In particular, $\tilde{\Gamma}$ is a simply connected $c.C_2$-geometry with point residues $H(3, 2^2)$. Note that there is a flag-transitive classical $c.C_2$-geometry Γ_0 on 378 points with point residues $H(3, 2^2)$, admitting $3U_4(3)$ (see [31] or [1]). Now, by coset enumeration using Cayley, we have $|\tilde{X} : \tilde{X}_a| = 378$. Thus $\tilde{\Gamma}$ is isomorphic to Γ_0 and $\tilde{X} \simeq 3U_4(3)$. Then any flag-transitive classical $c.C_2$-geometry is a quotient of Γ_0. The geometry Γ_0 has a unique proper quotient geometry on 126 points, since the central elements of order 3 acts fixed point freely on the set of points. Thus, we have proved:

THEOREM 23. *Assume that Γ is a flag-transitive classical $c.C_2$-geometry with $\Gamma_a \simeq H(3, 2^2)$. Then Γ is isomorphic either to the simply connected geometry on* 378 *points admitting* $3.U_4(3)$ *or to its quotient geometry on* 126 *points by the center of* $3U_4(3)$.

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References

1. A. Brouwer, A. Cohen and A. Neumaier, *Distance Regular Graphs,* Ergeb. Math. 3 Folge **18,** Springer-Verlag, Berlin, 1989.
2. F. Buekenhout and X. Hubaut, Locally polar spaces and related rank 3 groups, *J. Algebra,* **45** (1977), 391–434.
3. P. J. Cameron, D. Hughes and A. Pasini, Extended generalized quadrangles, *Geom. Ded.,* **35** (1990), 193–228.
4. A. Cohen and E. Shult, Affine polar spaces, to appear.
5. J. Conway, R. Curtis, S. Norton, R. Parker and R. Wilson, *Atlas of Finite Groups,* Clarendon Press, Oxford, 1985.
6. H. Cuypers, Finite locally generalized quadrangles with affine planes, preprint, Michigan State Univ., 1990.
7. A. Del Fra, D. Ghinelli, T. Meixner and A. Pasini, Flag-transitive extensions of C_n geometries, *Geom. Ded.* to appear.
8. B. Fischer, Finite groups generated by 3-transpositions, I, *Invent. Math.,* **13** (1971), 232–246. Also: Finite groups generated by 3-transpositions, Lecture Notes, Math. Inst. Univ. Warwick.
9. J. Goethals and J. Seidel, The regular two-graph on 276 vertices, *Discr. Math.,* **12** (1975), 232–246.
10. S. Heiss, A class of parabolic systems of rank 3 containing $\tilde{S}_6$, preprint, 1989.
11. A. A. Ivanov, On 2-transitive graphs of girth 5, *Europ. J. Combin.,* **8** (1987), 393–420.
12. A. A. Ivanov, Geometric presentations of groups with an application to the Monster, preprint, 1990.
13. W. Kantor, Generalized polygons, SCABs and GABs, in: *Buildings and the Geometry of Diagrams,* Lect. Notes Math. **1181**, (1986), Springer-Verlag, Berlin, 1986, pp. 79–158.
14. T. Meixner, Locally polar tower, preprint, 1989.
15. A. Pasini, Some remarks on covers and apartments, in: *Finite Geometries* (C. Baker and L. Batten, eds), Dekker, New York, 1985, pp. 223–250.
16. A. Pasini, A classification of a class of Buekenhout geometries exploiting amalgams and simple connectedness, University of Siena report no. 213, July 1989.
17. A. Pasini, A locally polar geometry associated with *HS, Europ. J. Combin.* to appear.
18. A. Pasini, On locally polar geometries whose planes are affine, *Geom. Ded.,* **34** (1990), 35–56.
19. S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles,* Res. Notes Math. **110,** Pitman, London, 1984.
20. M. Ronan, Coverings of certain finite geometries, in: *Ginite Geometries and Designs* (P. Cameron, J. Hirschfeld and D. Hughes, eds), Lond. Math. Soc. Lect. Notes Ser. **49,** Cambridge Univ. Press, 1981, pp. 316–331.
21. M. Ronan, *Lectures on Buildings,* Prespectives in Math. **7,** Academic Press, London, 1989.
22. M. Ronan and G. Stroth, Minimal parabolic geometries for the sporadic groups, *Europ. J. Combin.* **5** (1984), 59–91.
23. G. Seitz, Flag-transitive subgroups of Chevalley groups, *Ann. Math.,* **97** (1973), 27–56.
24. G. Stroth, Some geometry for *McL, Comm. Algebra,* **17**(11) (1989), 2825–2833.
25. M. Suzuki, Transitive extensions of a class of doubly transitive groups, *Nagoya Math. J.* **27** (1966), 159–169.
26. J. Tits, Non-existence de certaines extensions transitives I, Groupes projectifs a une dimension, *Bull. Soc. Math. Belg.* **23** (1971), 481–492.
27. J. Tits. Ensembles ordonnés, immeubles et sommes amalgamées, *Bull. Soc. Math. Belg.,* **A38** (1986), 367–387.
28. R. Weiss, A geometric characterization of the groups M_{22}, *He* and *Rud, J. Math. Soc. Japan,* to appear.
29. R. Weiss and S. Yoshiara, A geometric characterization of the groups *Suz* and *HS, J. Algebra,* **133** (1990), 182–196.
30. S. Yoshiara, A lattice theoretical construction of a GAB of the Suzuki sporadic simple group, *J. Algebra,* **112** (1988), 198–239.
31. S. Yoshiara, Some embeddings of flag-transitive classical locally polar geometries of rank 3, preprint, 1990.

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